Dynamic seismic ruptures on melting fault zones

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Abstract

We present a physical model which describes the behavior of spontaneous earthquake ruptures dynamically propagating on a fault zone and which accounts for the presence of frictional melt produced by the sliding surfaces. First, we analytically derive the solution for the temperature evolution inside the melt layer, which generalizes previous approximations. Then we incorporate such a solution in a numerical code for the solution of the elasto–dynamic problem. When a melt layer is formed, the linear slip–weakening law (initially governing the fault and relying on the Coulomb friction) is no longer valid. Therefore we introduce on the fault a linearly viscous rheology, with a temperature–dependent dynamic viscosity. We explore through numerical simulations the resulting behavior of the traction evolution in the cohesive zone before and after the transition from Coulomb friction and viscous rheology. The predictions of our model are in general agreement with data field from exhumed faults. We also find that the fault, after undergoing the breakdown stress drop controlled by the slip–weakening constitutive equation, experiences a second traction drop controlled by the exponential weakening of fault resistance due to the viscous rheology. This further drop enhances the instability of fault, increasing the rupture speeds, the peaks in fault slip velocity and the fracture energy density.

Key words: Earthquake Dynamics, Melting, Frictional Heat, Computational Seismology, Rheology of Faults and Constitutive Models.
1. Introduction

The melting process is a phase change of a substance from its solid state to the liquid state. The application of pressure or heat causes the internal energy of the substance to increase, resulting in a temperature rise up to the melting point, at which the solid undergoes to a less–ordered state (liquid phase).

Ice melting is familiar in skiing and it has recently considered a possible cause of Arctic warming (Screen and Simmonds, 2010). In rock mechanics it is now clear that the most of the energy dissipated on a seismogenic fault is ultimately converted into frictional heat (e.g., Pittarello et al., 2008) and that the temperature increase ($\Delta T$) due to seismic slip can exceed the melting temperature of crustal rocks.

Although rare field evidence for melting on exhumed faults engenders scepticism for the relevance of melt during earthquakes (Rempel and Rice, 2006), partial melting at local asperity contacts can occur (Jeffreys, 1942; McKenzie and Brune, 1972) and a continuous macroscopic melt layer may be present after some coseismic slip.

Indeed, large temperature increases leading to melting have been already obtained in dynamic models of spontaneously spreading earthquake ruptures obeying different fault governing laws. It has been found in numerical models that, for localized shear, both the thermal pressurization of pore fluids (Bizzarri and Cocco, 2006a, 2006b; henceforth BC06a and BC06b, respectively) and the flash heating of micro–asperity contacts (Bizzarri, 2009) do not reduce the frictional resistance on the fault surface enough to prevent melting. To date, the only possible exception (Bizzarri, 2010c) is represented by a slip– and velocity–dependent friction law, recently derived in high–velocity laboratory experiments by Sone and Shimamoto (2009).
On the other hand, evidence of melting has also been found in laboratory experiments, when conditions similar to those typical of seismic deformation are attained (e.g., Spray, 1995; Tsutsumi and Shimamoto, 1997; Hirose and Shimamoto, 2003).

When a continuous film of molten material is formed within the fault structure (see section 5.3), the “classical” governing models, essentially derived within the Coulomb–Amonton–Mohr framework, are no longer valid, since the coseismic increase in temperature affects the frictional properties of rocks (e.g., Sibson, 1977; Lachenbruch, 1980).

The main goal of the present paper is to extend previous spontaneous dynamic rupture models and to account also for non–Coulombian rheology of a fault. We will develop a physical model which, under some assumptions, incorporates the melt behavior (via a Newtonian rheology of a temperature–dependent viscous fluid). The present study also aims to extend previous papers (Nielsen et al., 2008, 2010), where a constant sliding velocity was assumed and where only the behavior after the onset of melting was explored. On the contrary, we account here for the transition from the “classical” behavior of rocks before melting to the viscous behavior after the formation of a melt layer.

The paper is organized as follows. In the next section we will describe the adopted fault model. The temperature evolution before the melting point is briefly summarized in section 3, while in section 4 we derive the time evolution of the temperature in the molten region (analytical details and comparisons with previous solutions are discussed in the appendixes). In section 5 we introduce the fault rheology (i.e., the fault boundary condition expressing the governing law). Section 6 is devoted to the introduction of the two–state physics (Stefan problem). The results of the numerical experiments on synthetic earthquakes, for a special case of melt layer evolution, are presented and discussed in sections 7 and 8. Section 9 discusses the
shape of the melt layer while the last section of the paper summarizes the prominent conclusions of the present study.

2. Model of the fault zone and statement of the problem

In the present paper we consider a more general fault structure than that adopted in BC06a and BC06b (see also Evans and Chester, 1995; Sibson, 2003 and Bizzarri, 2010a). As reported in Figure 1, a highly fractured, damage zone surrounds the slipping zone where the slip is concentrated. The latter can be regarded to represent the fault core, the ultracataclastic shear zone or the gouge layer. For simplicity we assume here that the slipping zone has the thickness $2w$, which is spatially homogeneous along the strike and the dip directions of the fault. The boundaries between the slipping zone and the damage zone are perpendicular to the normal fault coordinate, $\zeta$, which has its origin in the middle of the slipping zone. The plane $\zeta = 0$ can be associated with the principal slipping zone and can be regarded as the mathematical idealization of the fault surface (or fault plane) where the dynamic variables — such as traction, velocity, etc. — are formally defined.

Depending on the rupture dynamics, the frictional heat can be such that melting is produced. As a consequence a melt layer having thickness equal to $2w_m$ can also exist within the slipping zone of width $2w$ (Figure 1). By definition, melting occurs in a specific point if the temperature at that point exceeds the melting temperature, $T_m$. The resulting melt layer is also centered at $\zeta = 0$. This assumption is physically reasonable, since we consider spatially homogeneous properties within $2w$ (i.e., we neglect the chemical complexity of the minerals) and we know that in this case the maximum temperature is developed in the middle of the
slipping zone (let say, on the mathematical fault plane) and decreases for increasing off–fault distances (Andrews, 2002; BC06a). The thickness of melt layer increases through time, depending on the temperature evolution within $2w$, and its rate of increase, $(d/dt)w_m(t) ≡ \dot{w}_m(t)$, in full of generality can be variable through time. In the remainder of the paper, we will denote the left and right boundaries separating the solid and the melted rocks as $\zeta = -w_m(t)$ and $\zeta = w_m(t)$, respectively. These quantities, as well as $\dot{w}_m(t)$, are a priori unknown.

Since we presently do not have enough observational constrains to physically describe the physics of the damage zone, where elasto–plastic processes are expected to take place, in the model we will consider times up to time level at which the whole slipping zone has molten, i.e., we prescribe that $w_m ≤ w$. In other words we do not allow for the melting of the damage zone. Moreover, we do not account in the present model the melt removal by extrusion outside the slipping zone, through the so–called injection veins (Sibson, 1975).

In the following, for brevity of notation, we will omit the explicit dependence on the on–fault coordinates $\xi_1$ and $\xi_3$ while we only put the possible dependence on $\zeta$. We will denote with the symbol • the quantities pertaining to the melt layer and with $t_m$ the time instant when melting starts locally (i.e., at asperity contacts level). Time $t_m$ is formally defined by the following condition:

$$t_m \text{ such that } T(t_m) \equiv T(\zeta = 0, t = t_m) = T_m. (1)$$

Also $t_m$ is a priori unknown since it depends on the rupture dynamics (which in turn controls the temperature evolution, $T(\zeta, t)$, within the slipping zone).
Finally, in the following analysis we will assume that thermal pressurization is unimportant (i.e., we assume perfectly–drained configurations) and other weakening mechanisms are not operating. Of course, the inclusion of pore pressure variation can alter the dynamics of the fault (Andrews, 2002; BC06b) and ultimately the temperature developed by frictional heat.

### 3. Temperature evolution before the melting time

For times $t < t_m$ the whole slipping zone thickness is composed only by material (rocks, gouge, …) in the solid state; the temperature evolution is the solution of the heat conduction equation

$$\frac{\partial}{\partial t} T(\zeta, t) = \chi \frac{\partial^2}{\partial \zeta^2} T(\zeta, t) + \frac{1}{c} q(\zeta, t)$$  

(2)

where $\chi$ is the thermal diffusivity of the material in its solid state ($\chi = \kappa / \rho c_p$, where $\kappa$ is the thermal conductivity), assumed to be uniform along $\zeta$, $c \equiv \rho c_p$ is the heat capacity for unit volume of the bulk composite ($\rho$ being the cubic mass density of the composite and $C_p$ its specific heat at constant pressure) and $q$ is the heat generated for unit volume and for unit time ($[q] = J/(m^3 \text{ s}) = W/m^3$). Physically, $\chi$ expresses the ability of a substance to adjust its temperature to that of its surrounding (materials with a high value of $\chi$ conduct the heat quickly, compared to their volumetric heat, and therefore they rapidly adjust their temperature). $q$ represents the heat source due to frictional heat and its integral over the coordinate $\zeta$ gives the heat flux (i.e., the heat produced per unit area on the fault and per unit
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time). Equation (2) has the exact solution (see BC06a; their equation (A4)):

\[
T(\zeta, t) = T_0 + \frac{1}{4c \varepsilon w} \int_{0}^{\zeta-\varepsilon} d't' \left[ \text{erf}\left(\frac{\zeta + w}{2\sqrt{2}(t-t')}\right) - \text{erf}\left(\frac{\zeta - w}{2\sqrt{2}(t-t')}\right) \right] \tau(t')v(t')
\]  

(3)

where \( T_0 \) is the initial temperature distribution (i.e., \( T_0 = T_0(\zeta,0) \)), \( \text{erf}(\cdot) \) is the error function\(^1\) (\( \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_{0}^{\frac{\hat{z}}{2}} dx \ e^{-x^2} \)), \( 2w \) is the thickness of the slipping zone and \( \varepsilon \) is an arbitrarily small positive real number (see BC06a for further details). We assume here that \( 2w \) is spatially homogeneous and constant through time, although temporal variations in the slipping zone thickness can have relevant effects in the time scale of the seismic cycle of the fault (Bizzarri, 2010e). In equation (3) \( v \) denotes the magnitude of the fault slip velocity and \( \tau \) denotes the magnitude of the fault traction, expressed by the governing law in the unmelted regime, which can be the slip–weakening law (e.g., Ida, 1972), a rate– and state–dependent friction law (e.g., Dieterich, 1979), the law for the flash heating of micro–asperity contacts (Noda et al., 2009 and references therein), etc. (see Bizzarri, 2010a for a discussion). On the fault plane (i.e., in the limit \( \zeta = 0 \)) equation (3) reduces to (see also BC06a; their equation (6)):

\[
T^f(t) = T(0, t) = T_{0f}^f + \frac{1}{2c w} \int_{0}^{\zeta} d't' \ \text{erf}\left(\frac{w}{2\sqrt{2}(t-t')}\right) \tau(t')v(t')
\]  

(4)

where \( T_{0f}^f \equiv T(0, t) \), i.e., the initial temperature distribution on the fault plane. We simply recall here that solutions (3) and (4) pertain to the heat source

\[
q(\zeta, t) = \begin{cases} 
\frac{\tau(t)v(t)}{2w}, & t > 0, |\zeta| \leq w \\
0, & |\zeta| > w
\end{cases}
\]
which implicitly assumes that all the work spent to allow the fault sliding is converted into heat (see also Fialko, 2004; BC06a; BC06b; Pittarello et al., 2008; Bizzarri, 2009).

4. Temperature evolution after the melting point: behavior inside the melt layer

After the melting point $t_m$ we have a phase transition (from solid to molten materials) and therefore we have to consider a Stefan–like problem, accounting for two–state physics (see section 6). Here we will focus on the behavior of the temperature field inside the molten region. Specifically, we have to solve the following PDE:

$$\frac{\partial}{\partial t} \bar{T}(\zeta,t) = \bar{x} \frac{\partial^2}{\partial \zeta^2} \bar{T}(\zeta,t) + \frac{d}{dt} w_m(t) \frac{\partial}{\partial \zeta} \bar{T}(\zeta,t) + \frac{1}{c} \bar{q}(\zeta,t)$$

As discussed in detail by Nielsen et al. (2010), the term on the left hand side of equation (6) and the diffusion term $\frac{d}{dt} w_m(t) \frac{\partial}{\partial \zeta} \bar{T}(\zeta,t)$ can be neglected because they are dominated by the heat source term $\frac{1}{c} \bar{q}(\zeta,t)$ (as also checked numerically in Appendix B). Consequently, we have to solve the approximated equation

$$\frac{\partial}{\partial t} \bar{T}(\zeta,t) = -\frac{1}{c} \bar{q}(\zeta,t)$$

where $|\zeta| \leq w_m(t)$ and $t \geq t_m$. Let now consider the elementary, non singular, heat source
function

\[ \tilde{q}^{el}(\zeta, t) = \frac{h}{\sqrt{2\pi w_m(t)}} e^{-\frac{\zeta^2}{2w_m^2(t)}} \Theta(t - t_m) \]  

(8)

which has been frequently employed in the literature (e.g., Andrews, 2002; Noda et al., 2009) and assumes that the inelastic strain is distributed as a Gaussian in the distance \( \zeta \), with a standard deviation equal to the half–thickness of the melt layer (so that 68 % of the deformation occurs in a thickness of \( 2w_m \)). In equation (8) \( \Theta(.) \) is the Heaviside step function and \( [h] = \text{Pa/(m s)} \), so that \( [q^{el}] = \text{Pa/s} \). The elementary problem is completed by the boundary conditions:

\[ \tilde{T}(\zeta = -w_m(t), t) = T_m, \quad \forall t \geq t_m \]  

\[ \tilde{T}(\zeta = w_m(t), t) = T_m, \quad \forall t \geq t_m \]  

(9)

The elementary problem (equation (7) with (8)) has the following solution:

\[ \tilde{T}^{el}(\zeta, t) = C_1 + C_2 \zeta - \bar{h} \left( \frac{\zeta}{\sqrt{2 w_m(t)}} \right) + \frac{2}{\sqrt{\pi}} \frac{e^{-\frac{\zeta^2}{2w_m^2(t)}} w_m(t)}{2 \bar{c} \bar{\chi}} \]  

(10)

with \( t \geq t_m \). Note that, formally, the time dependence in \( \tilde{T}^{el} \) is implicit (it is due to time variability of \( w_m \)). In (10) the two constants of integration \( C_1 \) and \( C_2 \) are determined by considering the boundary conditions (9) that lead to:
where again \( t \geq t_m \). Simple algebra shows that (11) satisfies both conditions in (9), for arbitrary value of half–layer thickness (or in other words for all arbitrary times \( t \geq t_m \)). We remark that the solution (11) depends on \( w_m(t) \), which is still unknown; here we simply note that we have the following condition for \( w_m(t) \):

\[
\quad w_m(t) = 0, \quad \forall \quad t \leq t_m
\]  

(12)

stating the obvious fact that the thickness of the melt layer is null at the melting instant and does not exist before that time.

Let now we consider the heat input \( h = \bar{\tau}(\zeta, t) \nu(t) \); this gives the actual heat source:

\[
\quad q(\zeta, t) = \frac{\bar{\tau}(\zeta, t) \nu(t) e^{-\frac{\zeta^2}{2 w_m(t)}}}{\sqrt{2\pi} w_m(t)} \Theta(t-t_m)
\]  

(13)

In (13) \( \bar{\tau}(\zeta, t) \) is the traction when \( t \geq t_m \), which is described with more details in the next section. By using (13), the general evolution of the temperature inside the melt layer can be expressed as follows:
\[ T(\zeta, t) = T_m - \left[ \frac{\zeta \text{erf}\left(\frac{\zeta}{\sqrt{2} w_m(t)}\right)}{2 \tilde{c} \tilde{\chi}} \right. \\
\left. - \left( \sqrt{\frac{2\pi}{\tilde{c}^2}} + \pi \text{erf}\left(\frac{1}{\sqrt{2}}\right) - \sqrt{2\pi} e^{-\frac{\zeta^2}{2 w_m^2(t)}} \right) w_m(t) \right] \tilde{\tau}(\zeta, t)v(t) \]  

(14)

where again \( t \geq t_m \). Interestingly, we can note that at \( t = t_m \) the second term inside the square brackets vanishes (because of (12)), as does the first term too (we recall that (14) holds for \( \zeta \) in the interval \([-w_m(t), w_m(t)]\), which at \( t = t_m \) simply reduces to \( \zeta = 0 \), so that \( T(0, t_m) \equiv \bar{T}(t_m) = T_m \), in agreement with (1).

In the limit \( \zeta = 0 \) (exploiting again the condition (12)) equation (14) can be written as:

\[ \bar{T}(t) = T(0, t) = T_m + \left( \sqrt{\frac{2\pi}{\tilde{c}^2}} + \pi \text{erf}\left(\frac{1}{\sqrt{2}}\right) - \sqrt{2\pi} \right) \Theta(t - t_m) w_m(t) \tilde{\tau}(t)v(t) \]  

(15)

5. Fault rheology

5.1. Coulomb friction before melting

In the previous two sections we have invoked the shear traction \( \tau \) for the phase prior to melting (section 3) and \( \bar{\tau} \) after the melting instant (\( t_m \)). In this section we will discuss in more
details how to incorporate the fault rheology (i.e., the fault governing law) in our dynamic model.

For $t < t_m$ the quantity $\tau$ is expressed by one of the “classical” friction laws based on the Amonton–Coulomb–Mohr theory, stating a linear proportionality between $\tau$ and effective normal stress $\sigma_n^{\text{eff}}$, through the friction coefficient $\mu$ ($\tau = \mu \sigma_n^{\text{eff}}$). For simplicity and to better understand the effects of the presence of melting we adopt here the widely adopted linear slip–weakening (SW henceforth) constitutive relation, recalled here for convenience:

$$\tau^{(\text{SW})} = \begin{cases} 
\tau_u - (\tau_u - \tau_f) \frac{u}{d_0}, & u < d_0 \\
\tau_f, & u \geq d_0
\end{cases}$$

where $\tau_u$ defines the upper yield stress ($\tau_u = \mu_u \sigma_n^{\text{eff}}$), $\tau_f$ defines the upper kinetic stress ($\tau_f = \mu_f \sigma_n^{\text{eff}}$) and $d_0$ is the characteristic SW distance, quantifying the amount of cumulative slip required to complete the breakdown process (i.e., the stress release).

5.2. Governing law for a continuous melt layer

In the presence of melting the relation $\tau = \mu \sigma_n^{\text{eff}}$ is no longer valid, due to a more complex coupling between traction and normal stress. Nielsen et al. (2008; their equation (57)) found an approximate behavior for $\tau$ at high slip rates (namely, for $v \gg W = \sqrt{\frac{8 \tilde{T}_c \tilde{\chi} \tilde{c}}{\eta_c}} \approx 0.4$ m/s for typical parameters; see Table 1). However, this approximation holds only in the special case of steady state motion, and not in the case of variable slip velocity as in spontaneous rupture models.
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Following Fialko (2004), in a molten film of width $2w_m$ most of the resistance to slip comes from viscous deformation of the molten layer. By assuming a Newtonian fluid, postulating a linear dependence between the applied stress and the resulting rate of shear strain $\dot{e}$, we have:

$$\tau^{(NF)} = \eta \dot{e}$$

(17)

where $\eta$ is the dynamic viscosity of the melt material ($\eta = \eta(\zeta,t)$; note that the dependence of $\eta$ on $\zeta$ and $t$ is implicit, since $\eta = \eta(\tilde{T})$ and $\tilde{T}$ depends on $\zeta$ and $t$). Several studies (e.g., Shaw, 1972; Dingwell, 1998) indicate that the temperature dependence of viscosity can be satisfactorily described by the Arrhenius law

$$\eta(\zeta,t) = \tilde{K} e^{\frac{\tilde{T}_a}{(\tilde{T}(\zeta,t) - \zeta) + 273.15}}$$

(18)

where $\tilde{K}$ is empirical constant ($[\tilde{K}] =$ Pa s) and $\tilde{T}_a$ is the activation temperature ($\tilde{T}_a = \tilde{E}_a / R$, being $\tilde{E}_a$ the activation energy and $R$ the universal gas constant; see also Fialko and Khazan, 2005). Both the constants $\tilde{K}$ and $\tilde{T}_a$ depend on the rock composition. In (18) the temperature has to be expressed in K (as $\tilde{T}_a$) so we apply to $\tilde{T}$ the usual shift (because $[\tilde{T}] = ^\circ C$). Moreover, we note that in (18) the melting temperature is computed at the previous time level and therefore it enters into the heat source (13) as a known quantity. We will discuss in more details this approximation in Appendix C. On the other hand, we can express the deformation rate in the limit $\zeta = 0$ as:
\[
\dot{\gamma} = \frac{v}{2w_m}
\]

so that (17) becomes (see also Fialko, 2004):

\[
\tau^{(NF)} = \eta \frac{v}{2w_m}
\]

Finally, on the fault plane, by combining (20) and (18), we obtain:

\[
\tau^{(NF)}(t) = \tilde{\eta} e^{\tilde{T}/(T_c - \epsilon) + 273.15} \frac{v(t)}{2w_m(t)}
\]

where \(\tilde{T}/\) is given by equation (15). In the limit of isoviscous melt (i.e., if we neglect the temperature dependence of viscosity), we will simply have:

\[
\tau^{(NF)}(t) = \tilde{\eta}_m \frac{v(t)}{2w_m(t)}
\]

where \(\tilde{\eta}_m = \tilde{\eta}(\tilde{T} = T_m) = K e^{\tilde{T}/(T_m + 273.15)}\) is the reference dynamic viscosity at the melting point. Equations (20) and (22) can be regarded as the simplest case of a viscous fault rheology, which in general can be expressed as \(\tau^n = v \alpha_n\), where \(n\) is a constant and \(\alpha_n\) effectively controls the strength of the fault (see also Hetland et al., 2010); when \(n = 1\) (linear viscous rheology), \(\alpha_1 = \tilde{\eta}_m/(2w_m)\).
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5.3. Transition from frictional resistance to viscous shear

First of all we notice that the frictional resistance $\tau^{(NF)}$ given by (21) (or by (22)) for small values of $w_m$ can be greater than the average Coulomb–Mohr failure stress for the upper crust and greater than $\tau^{(SW)}$ expressed by (16). This would imply that, immediately after $t_m$, the fault will experience a significant increase of resistance to slip, which in turn can stop the ongoing rupture. Indeed, experiments by Tsutsumi and Shimamoto (1997) suggest that viscous braking might result after melting. Moreover, when the onset of melting is accompanied by increases in shear stress exceeding the static friction, or the intrinsic rock strength, the fused fault may be abandoned, and the slip may be transferred to a new subparallel plane (e.g., Swanson, 1992; Otsuki et al., 2003). As a consequence, it is possible than viscous braking causes the formation of multiple melt layers. These phenomena still require further observational constraints and we are not able to fully include in the model. Therefore we make the conservative assumption that, after $t_m$, $\tau$ is still described by Coulomb friction until the melt layer is sufficiently thick ($w_m = w_{mc}$), so that $\tau^{(NF)}$ is lower (dominant) with respect to the Coulomb friction. This condition physically defines the formation of a continuous melt layer (having an initial width of $2w_{mc}$). The macroscopic, continuous melt layer trapped between the fault walls, as opposed to microscopic melting occurring at asperity contacts level (at time $t_m$), would take place after some cosesimic slip. In many field observations this continuous layer can be absent due to processes such as melt extrusion, not considered here.

By considering that $\tau$ still follows a Coulomb friction until $w_m = w_{mc}$, we guarantee a continuous spreading of the propagating rupture; we also define an effective melting time $t_m^{\text{eff}}$.  

at which temperature exceeds an effective melting temperature $T_m^{\text{eff}} > T_m^*$:

$$T_m^{\text{eff}} \equiv T(t_m^{\text{eff}}) = T(0,t_m^{\text{eff}}) \text{ such that } \tau^{(\text{NF})}(t_m^{\text{eff}}) < \tau^{(\text{SW})}(t_m^{\text{eff}})$$

(23)

Practically, provided that $w_m(t)$ is determined — see next sections — we evaluate the fault temperature as follows:

$$
\begin{align*}
T & = T^f + \frac{1}{2 \epsilon w} \int_0^{t_m^{\text{eff}}} dt' \text{ erf} \left( \frac{w}{2 \sqrt{\chi (t - t')}} \right) \tau(t') \nu(t') \\
\bar{T} & = T_m + \frac{\left( \sqrt{\frac{2\pi}{\epsilon}} + \pi \text{ erf} \left( \frac{1}{\sqrt{2\pi}} \right) - \sqrt{2\pi} \right) \bar{w}(t) \bar{\tau}(t) \nu(t)}{2 \pi \bar{c} \bar{\chi}} \\
\end{align*}
$$

(24)

with $\tau^{(\text{SW})}$ as in (16) and $\tau^{(\text{NF})}$ as in (21) or (22).

We finally note that the adoption of the SW law enables us to identify the time instant when viscous shear takes over ($t_m^{\text{eff}}$) and correspondently the value of $T_m^{\text{eff}}$ and the critical melt layer half–thickness $w_{m_c}$.

**6. The Stefan problem**

In the previous two sections $w_m(t)$ appears as an unknown quantity, but it is necessary to evaluate the fault temperature (equation (24)) and fault traction (equations (21) or (22)). As mentioned above, $w_m(t)$ is determined by considering the Stefan problem, which reads:
in which \( t \geq t_m \) and \( L \) is the latent heat of fusion. Equation (25) expresses the balance between the heat \( dQ \) required to change state (i.e., to melt) of a rock mass \( dm \) within the time \( dt \) (\( dQ = L \, dm = \rho L \, dV = \rho L \, d\xi_1 \, d\xi_3 \, dw_m \)) and the Fourier heat flux through the melt–solid boundary \( \zeta = w_m(t) \) (\( q^+ - \bar{q}^- = c \, \chi \, \frac{\partial}{\partial \zeta} \tilde{T}(\zeta,t) \bigg|_{\zeta = w_m^-(t)} - \tilde{c} \, \chi \, \frac{\partial}{\partial \zeta} \tilde{T}(\zeta,t) \bigg|_{\zeta = w_m^-(t)} \)); all the absorbed energy goes into the phase change (from solid to liquid), without affecting the temperature in the surroundings. The second term on the right hand side of equation (25) can be obtained from (14), which we recall is the solution of the approximation (7) of equation (6). Note also that (14) would also require a physical model to describe \( \tau \) for all \( \zeta \) and not only on the fault plane; this model requires observational constraints that are presently missed and further investigations. On the other hand, the first term on the right hand side of (25) cannot be calculated from equation (3), since that solution holds only in the phase prior to melting; on the contrary, for \( \zeta > w_m(t) \), it can be obtained by solving the following diffusion problem:

\[
\frac{d}{dt} w_m(t) = -\frac{1}{\rho L} \left\{ \frac{\partial}{\partial \zeta} \tilde{T}(\zeta,t) \bigg|_{\zeta = w_m^-(t)} - \tilde{c} \frac{\partial}{\partial \zeta} \tilde{T}(\zeta,t) \bigg|_{\zeta = w_m^-(t)} \right\} \quad (25)
\]

\[
\begin{align*}
\frac{d}{dt} w_m(t) = 0, \forall t \leq t_m \\
\frac{\partial}{\partial \zeta} \tilde{T}(\zeta,t) \bigg|_{\zeta = w_m^-(t)} - \tilde{c} \frac{\partial}{\partial \zeta} \tilde{T}(\zeta,t) \bigg|_{\zeta = w_m^-(t)} , \forall t \geq t_m
\end{align*}
\]

In (26) the spatial coordinate \( \xi \) quantifies the distance from the moving melt–solid boundary and is related to \( \zeta \) used above through the relation \( \xi = \zeta - w_m(t) \) (see Figure 1) and \( \hat{q} \) indicate possible additional heat sources or sinks. Similar equations in \( \xi' = \xi - w_m(t) \)
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give \( \dot{T} \) for \( \zeta < - \omega_m(t) \). (The use of symbol \( \dot{T} \) in (25) and (26) emphasizes that the
temperature is not the same as in equation (3), because we are now dealing with a
two–phase problem.)

The problem (26) can be treated analytically by using the Laplace transform, which gives a
subsidary 2nd order ODE in \( \zeta \) for the Laplace–transformed function \( \tilde{T}(\zeta, \omega) \), which depends
on the temporal frequency \( \omega \). Even in the absence of heat sources or sinks \( (\dot{q} = 0) \) the function
\( \tilde{T}(\zeta, \omega) \) does not admit a closed–form inverse Laplace transform, so that the analytical solution
for \( \dot{T}(\zeta, t) \) (and for \( \dot{T}(\zeta, t) \)) remains implicit. As a consequence, problem (26) has to be solved
numerically; this would be the matter of a future study.

7. Numerical results

It is reasonable that the solution \( w_m(t) \) of equation (25) is a sufficiently regular, real
function. We can then expand it in a Taylor series in \( t \) with initial point in \( t_m \) as follows:

\[
\begin{align*}
    w_m(t) &= w_m(t_m) + \frac{d}{dt}w_m(t) \bigg|_{t=t_m}(t-t_m) + \frac{1}{2} \frac{d^2}{dt^2}w_m(t) \bigg|_{t=t_m}(t-t_m)^2 + \ldots.
\end{align*}
\]

In the remainder of the paper we will consider the first–order approximation of \( w_m(t) \); taking into account the
condition (12) we have, for \( t \geq t_m \):

\[
w_m(t) \approx \dot{w}_m(t - t_m)
\]  (27)
where \( \dot{w}_m \equiv \frac{d}{dt} w_m \bigg|_{t=t_u} \). We will consider different configurations, by attributing to \( \dot{w}_m \) values suggested by observations. We emphasize that this is not the exact solution of the problem (25) coupled with (26), but it is its first–order approximation. The conservative assumption that the melt layer enlarges at a constant rate makes the problem tractable analytically and can help us in quantifying the prominent effects of the molten material on the dynamics of a fault. Equation (27) can be thus regarded as a proxy of the true enlarging behavior of the melt layer.

The fully dynamic, spontaneous rupture problem is solved via the finite difference code described in Bizzarri and Cocco (2005). To minimize the spurious numerical reflections originating from the domain boundaries we apply the absorbing boundary conditions described in Bizzarri and Spudich (2008). The rupture nucleates in the imposed hypocenter \( H \) (see Figure 2), located at a depth \( x_3^H = 7 \) km, and propagates at early states at a constant rupture speed (2.4 km/s; see Bizzarri, 2010b) and then in a fully spontaneous fashion; the fault slips forever, until unbreakable barriers (located the bottom and left ends of the fault) are reached. The adopted parameters are reported in Table 1.

In Figure 3 we report a synoptic comparison between a numerical simulation obeying the linear SW law (16) over the whole time window considered (left column) and another simulation where the effects of melting are accounted for (right column). Due to the symmetry exploitation along the strike direction (Bizzarri, 2009) we plot only the one half of the fault. When melting is included into the model we consider \( \dot{w}_m = 0.5 \) mm/s in equation (27) (a value also suggested by laboratory observations of Nielsen et al., 2008, 2010) and the temperature is calculated from equation (24). The adopted frictional parameters guarantee that the rupture
becomes supershear (Figure 3a). However, when a viscous rheology is considered the transition to supershear regime occurs earlier and there are larger patches on the fault with $v_r > v_S$ (Figure 3b; $v_r$ is the rupture speed and $v_S$ is the $S$ waves speed). This indicates that the transition to a viscous rheology as that assumed here enhances the fault instability and therefore promotes the supersonic regime. Correspondently, peaks attained by the fault slip velocity are larger in the case of viscous rheology (compare Figures 3c and 3d). Moreover, the value of the seismic moment at the end of the numerical experiments is rather different; $M_0 = 3.01 \times 10^{18}$ Nm ($M_w = 6.3$) in the reference (i.e., Coulombian) case, while $M_0 = 1.66 \times 10^{19}$ Nm ($M_w = 6.8$) in the viscous case.

Figure 4 shows the solutions for the simulation reported in Figures 3b and 3d in a target fault node. In that location the melting point is reached at $t = t_m = 0.86$ s; after $t_m$, $w_m$ evolves accordingly to (27) and $\tilde{\eta}$ follows equation (18). The fault experiences a first traction drop, which namely is the breakdown stress drop, $\Delta \tau_b = \tau_u - \tau_f = 26.04$ MPa. The traction remains at $\tau_f$ for a while and then, at $t = t_{m}^{\text{eff}} = 0.93$ s (see equation (23)), the time evolutions of temperature, fault slip velocity, dynamic viscosity and melt layer thickness are such that the viscous shear is dominant with respect to the SW law (16). According to section 5.3, after $t_{m}^{\text{eff}}$ the fault traction is described by a linearly viscous rheology (equation (21)), emphasized by labels in Figure 4. Due to the decrease in $\tilde{\eta}$ (caused in turn by the temperature increase after $t_{m}^{\text{eff}}$, see Figure 4b), the traction exhibits a second drop (Figure 4c), which is roughly twice of $\Delta \tau_b$. The total drop is then roughly equal to 80 MPa, which is compatible with observations (see Figure 10 in Nielsen et al., 2010, where the dependence of the traction drop on applied...
normal load is shown). In this numerical experiment the final value of traction is 1.85 MPa (for a Coulomb rheology this would correspond to a friction coefficient equal to 0.015).

The traction vs. slip curve (Figure 5) shows a sufficiently good agreement with field data collected on an exhumed seismic thrust fault zone in Outer Hebrides, Scotland (surveyed in 2005 and having a focal depth roughly equal to 10 km; Hirose, 2005, unpublished data) and with measurements from Sibson (1975) performed on the same fault zone (open and full blue circles, respectively). We can roughly estimate the value of viscous shear as (cfr. Di Toro et al., 2006):

$$\langle \ddot{x} \rangle \approx \frac{2 \bar{w}_m E \bar{\rho}}{u - u(t_m^{\text{eff}})}$$  \hspace{1cm} (28)

where $\bar{w}_m$ is the average value of melt layer thickness ($\bar{w}_m = \int_{t_{\text{eff}}^{\text{end}}}^{t_{\text{end}}} \dot{w}_m(t) \, dt$, which in the present simulation equals $\dot{w}_m(t_{\text{end}} - t_m^{\text{eff}})$, $t_{\text{end}}$ being the final time of the computation) and $E$ is the energy required to produce 1 kg of melt. From equation (28), by assuming $E = 1.76$ MJ/kg and $\bar{\rho}$ as in Table 1 and considering that $\bar{w}_m = 1$ mm and that $u(t_m^{\text{eff}}) = 0.56$ m, we obtain the red curve plotted in Figure 5, which is in general agreement with the grey curve in Figure 5 (which in turn represents the result of our model in the viscous regime).

8. The importance of the melt layer evolution

In this section we will explore the effects of different temporal evolutions of the melt layer thickness $2w_m$, by assuming enlarging rates compatible with observations (Nielsen et al., 2008;
their Table 4). \( w_m \) still obeys (27), with the condition \( w_m \leq w \). (Again, we recall here that equation (27) is the first–order approximation of the solution of equations (25) and (26).) We consider two different configurations, having \( 2w = 2 \) mm and \( 2w = 14 \) mm, values representative of the slipping zone thickness. The results are reported in Figures 6 and 7, respectively, where we also superimpose the reference solution, i.e., the simulation in which melting effects are not considered in the model (black curves in all panels).

It is clear that the time evolution of \( w_m \) controls \( t_m^{\text{eff}} \), i.e., the instant when rheology departs from Coulomb friction and becomes viscous (see section 5.3). In particular, as \( \dot{w}_m \) increases the fault remains at the kinetic friction level \( \tau_f \) for less time (Figures 6a and 7a) and for smaller amount of cumulative slip (Figures 6c and 7c). In the extreme case (\( w = 1 \) mm and \( \dot{w}_m = 5 \) mm/s; yellow curves in Figure 6) the transition between Coulomb and viscous behavior occurs within the breakdown zone (i.e., for slips smaller than \( d_0 \)); in this case there are no longer two separate drops in traction, but the fault weakening is continuous.

Correspondingly, the enhanced stress drop at a specific fault point causes a stress redistribution in its surroundings and this increase of dynamic load ultimately causes an increase of rupture speed. This can be clearly seen in Figures 6a and 7a, since the rupture time at the target fault location decreases as \( \dot{w}_m \) increases. The same holds for peaks in fault slip velocity; it increases as \( \dot{w}_m \) increases (Figures 6d and 7d). Especially in the case of localized shear (\( w = 1 \) mm), a faster increasing rate of the melt layer causes a shorter breakdown zone time (which is the time required for traction to drop from \( \tau_u \) down to \( \tau_f \); Bizzarri et al., 2001).

With the only exception of \( \dot{w}_m = 5 \) mm/s (which is in fact larger than values observed experimentally; see Nielsen et al., 2008), all models with melting effects develop reasonable
temperatures; on the contrary, the case which does not correctly model melting effects (black curves) would produce arbitrarily large temperatures; when \( w = 1 \) mm the temperature predicted by the model at the end of the simulation is larger than the average temperature estimated for the Earth’s core. A typical temperature distribution on the mathematical fault plane is reported in Figure 8 for the case of a slipping zone 2 mm wide and for \( \dot{w}_m = 0.1 \) mm/s. A fault node of the unbroken region (at rest) remains at \( T_{f0} \) until rupture front reaches it; then temperature increases, exceeds \( T_m \) and then it is controlled by the values of traction and fault slip velocity (see equation (15)).

9. On the shape of the melt layer

The temperature evolution in a fault node depends on the traction and slip velocity histories in that point, as expressed by equation (4). As a consequence, the melting instant \( t_m \) would be in fact the two–dimensional array \( t_m(x_1, x_3) \). To better quantify this, we plot in Figure 9a the spatial distribution of melting instants in the case of \( w = 1 \) mm and \( \dot{w}_m = 0.1 \) mm/s. We can clearly see that minimum value of \( t_m \) array (which in turn defines the first fault node where melt occurs) is attained at the imposed hypocenter \( H \), which by definition is the first point undergoing to instability. We remark here that for homogeneous rheology and constant and spatially uniform \( \dot{w}_m \) the heat production rate is such that the melting temperature is always reached first in \( H \) (in other words \( t_m \) is minimum in \( (x_1^H, x_3^H) \)). The shape of the melt layer half–thickness, as given by equation (27), is reported in Figure 9b, from which we can see that \( w_m \) is maximum in \( H \) (where the term \( t - t_m \) is maximum).
In Figure 10 are reported the increasing histories of $w_m$ for two representative values of $\dot{w}_m$. Each line represents the shape of $w_m$ as a function of the depth, calculated at 3 km from the hypocenter (along the strike direction) and every 0.1 s. These profiles confirm that the maximum extension of the melt layer is at hypocentral depths. While for moderate growth rates $w_m$ is quite small with respect to $w$, when $\dot{w}_m$ is sufficiently high (for instance $\dot{w}_m = 1$ mm/s as in Figure 10b), it might happen that at the hypocenter $w_m$ exceeds $w$, which represents the upper limit for our model (as discussed above; see section 2). We emphasize that the shape of the melting layer, during its evolution, depends on the imposed value of $\dot{w}_m$ appearing in equation (27), but it is also controlled by the temperature evolution (which determines $t_m(x_1,x_3)$), which in turn depends on the fault dynamics.

10. Discussion and conclusions

In this paper we have presented a physical model to account for rocks melting during coseismic earthquake ruptures spontaneously spreading on a fault of finite width, by considering Coulomb friction and viscous rheology in one framework. We have solved the equations of heat transfer in presence of melting, and we have incorporated such a solution in a numerical code to solve the elasto–dynamic problem. Our solution is in agreement and generalizes previous studies where a constant heat input was considered (Fialko and Khazan, 2005; Nielsen et al., 2008, 2010).

In our model we have made some assumptions, briefly recalled here. i) We require that the melt layer can reach, at maximum, the boundary of the slipping zone thickness, but it can not affect the surrounding damage zone (see Figure 1). This assumption is reasonable, since field
data reported by Nielsen et al. (2010) indicate that the thicknesses of melt layer typically are of the order of a fraction of millimeter, with a few exceptions reaching several millimeters. 

*ii*) We neglect extrusion dynamics, i.e., we do not consider the formation of the injection veins (Sibson, 1975). At the present state of knowledge we do not have sufficient information to analytically model the extrusion process in natural faults, if any (Sirono et al., 2006). 

*iii*) Due to the small temporal scale pertaining to the breakdown process, during which the stress release takes place, we can safely assume that the temperature inside the melt layer remains well above the melting temperature, $T_m$, so we can neglect the melt solidification process. This would become potentially important in the post–seismic phase of the dynamic rupture, not considered here. 

*iv*) We have also neglected the phenomenon pre–melting (or surface melting), describing the fact that a quasi–liquid layer (which is in turn temperature–dependent) can appear on crystalline surfaces, even below the $T_m$. 

*v*) Most earthquakes happen along faults that contain a range of mineral compositions; for simplicity we have considered here a single value of $T_m$, which has to be regarded as an average, representative quantification of the melting temperature of the material assemblage in the slipping zone. This is reasonable, in that the boundary between solid and melt appears quite well defined in most laboratory samples and samples from natural faults. 

*v*) The fault initially obeys the linear SW law and then is governed by a viscous rheology. The transition between a Coulomb rheology to a viscous rheology occurs spontaneously, as discussed in section 5.3, and depends on the evolution of the temperature on the fault surface. We emphasize that the model proposed here can be generalized to other more elaborated Coulombian governing models, such as non–linear SW equations or rate– and state–dependent friction laws (see Bizzarri, 2010a for a review). The adoption of a linear SW friction before melting makes simple the identification of the
transition to viscous rheology. vii) We assume an analytical time evolution of the melt layer thickness $w_m(t)$; it is a first–order approximation of the true behavior of the growing melt layer, which can be obtained only numerically by solving the coupled equations (25) and (26).

Given all the above–mentioned limitations of the present model we are able to explore the behavior of a dynamically propagating rupture above the melting temperature ($T_m$). Otherwise we would have been forced to stop the numerical simulation when $T_m$ was reached in a fault node. Previous theoretical studies clearly indicate that $T_m$ (Bizzarri and Cocco, 2006a, 2006b; Fialko, 2004; Bizzarri, 2009 and references therein) can be easily exceeded, independent of the adopted constitutive equation, provided that the shear is sufficiently localized ($w \leq 1$ mm for representative values of the effective normal stress).

A prominent outcome of the present model is that after melting, the fault experiences a second traction drop which can be twice (or more) the breakdown stress drop predicted by the simple linear SW law (see Figures 4b, 6a and 7a). Correspondingly, the fracture energy density — which is the as the amount of energy (for unit fault surface) necessary to maintain an ongoing rupture which propagates on a fault; see Bizzarri (2010d) and references cited therein — increases. This is a consequence of the conservative choice we made, that no viscous braking can occur after melting (see section 5.3).

We also found that the supershear regime is promoted by the transition to a viscous rheology and this can have significant effects on the resulting ground motions (Dunham and Bhat, 2008; Bizzarri et al., 2010). We emphasize that all the previous features are preserved varying the value of the enlarging rate of the melt layer.

We note that the traction during the viscous stage of the rupture predicted by our model exhibits an exponential decay with time, as early postulated by Lachenbruch (1980),
theoretically derived by Matsu’ura et al. (1992), observed in laboratory experiments by Ohnaka and Yamashita (1989), and corroborated by the more recent high–velocity experiments by Sone and Shimamoto (2009; see also Bizzarri, 2010c). The viscous behavior we model is also in satisfactory agreement (see Figure 5) with field data from an exhumed seismic thrust fault zone (Sibson, 1975; Hirose, 2005, unpublished data).

The weakening rate in the viscous regime and the duration of the second traction drop are primarily controlled by the time evolution of the melt layer thickness. The latter can be obtained by solving numerically equations (25) and (26), which surpasses the purposes of the present study. In fact, this would require a consistent physical model to fully describe the behavior of $\tau$ for all the coordinates $\zeta$ and not only in the mathematical fault plane ($\zeta = 0$), but this needs further observational constraints. Here we have adopted a first–order approximation of the function $w_m(t)$, as in equation (27), which makes the problem tractable analytically and overcomes the previous theoretical problem.

Further development of this work may be the comparison between our theoretical predictions and high–velocity friction experiments, conducted with time–variable slip velocity histories compatible with those obtained in dynamic models and normal loads representative of seismogenic depths. Finally, systematic microstructural analysis of rock samples can potentially illuminate us about the chemical complexity of natural faults.

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Appendix A. Properties of the temperature inside the melt layer

A.1. Comparison with previous solution

In this appendix we will analyze the temperature evolution within the molten region. As discussed in the main text, the analytical solution for $\tilde{T}$ is given by equation (14), which reduces to equation (15) in the center of the melt layer (i.e., on the mathematical fault plane $\zeta = 0$; see Figure 1).

Nielsen et al. (2008) (and Nielsen et al., 2010 as well), solve the heat conduction equation (6) in a static (i.e., time independent) configuration. Their solution can not be expressed in a closed form as a function of $w_m$, so they apply an approximation of boundary condition for the flux at the melt–solid boundary:

$$\left. \frac{\partial \tilde{T}}{\partial \zeta} \right|_{\zeta = w_a} \approx - \frac{\tau_a v_a}{2\tilde{c}\tilde{\xi}} \quad (A1)$$

where $\tau_a$ is the constant applied stress and $v_a$ is the applied sliding velocity (see equation (16) in Nielsen et al., 2008; note also that in our notation $\tilde{c}$ is equivalent to $\tilde{\rho}\tilde{c}$ in the notation of Nielsen et al., 2008). Correcting a misprint in their equation (B2), they obtain:

$$\tilde{T}^{NEA} (\zeta) = T_m - T_c \log \left( \cosh^2 \left[ \frac{2\zeta \tau_a}{\tilde{\eta}_c W} \sqrt{\frac{v_a^2}{W^2} + 1} \right] \right) \quad (A2)$$

(see also equations (34) and (35) in Nielsen et al., 2008). Simple algebra shows that the
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function $\tilde{T}^{\text{NEA}}(\zeta)$ does not satisfy the boundary condition $\tilde{T}^{\text{NEA}}(\pm w_m) = T_m$ for arbitrary values of melt layer half–thickness, but only if $w_m$ equals a specific value, $w_m^*$:

$$w_m^* = \frac{W \bar{\eta}_m}{2 \tau_a} \begin{pmatrix} \text{atgh} \left( \frac{v_a}{W \sqrt{\frac{v_a^2}{W^2} + 1}} \right) \end{pmatrix}$$

Let we now consider our solution for $\tilde{T}^f(\zeta,t)$ (see equation (14)) and let us consider the temporal averages (for times $t \geq t_m$) of time variable quantities appearing therein; we can write:

$$\overline{T}(\zeta) = T_m - \begin{pmatrix} \frac{\zeta}{\sqrt{2 \pi w_m^2}} \end{pmatrix} \begin{pmatrix} \text{ erf} \left( \frac{2\pi}{\sqrt{e}} \frac{1}{\sqrt{2}} - \sqrt{2\pi} e^{-\frac{\zeta^2}{2\pi w_m^2}} \right) \end{pmatrix}$$

If we associate the quantities $\overline{T} \leftrightarrow \tau_a$ and $\overline{v} \leftrightarrow v_a$ we can directly compare the temperature distribution as a function of the distance from the center of the melt layer obtained here (equation (14), or its time average, equation (A4)) with the solution of Nielsen et al. (2010; see equation (A2)).

On the other hand, Fialko and Khazan (2005; their equation (22)), in the isoviscous approximation independently found another expression for $\tilde{T}^f(\zeta,t)$:
\[
\tilde{T}^{fK}(\zeta) = T_m + \frac{v_a^2 \tilde{\eta}_m}{2 \tilde{\chi}} \left( 1 - \frac{\zeta^2}{w_m^*} \right)
\]  

(A5)

The result of the comparison between the three different equations ((14) or its time average (A4), (A2) and (A5)) is reported in Figure A1. We select three representative values of the sliding speed, \(v_a = 0.1\) m/s (Figure A1a), \(v_a = 1\) m/s (Figure A1b) and \(v_a = 10\) m/s (Figure A1c). Note that to make possible the comparison we use in (A4) \(\bar{w}_m = w_m^*\) as given by equation (A3), since it is the only value which satisfies condition (9) for the solution of Nielsen et al. (2008, 2010). Analogously, in equation (A5) we also use \(w_m = w_m^*\).

We can clearly see that in all three models the maximum temperature is realized in the center of the melt zone (as physically expected) and it gradually decreases near the melt–solid boundaries. For decreasing melt layer thickness the temperature values are higher, as they should be, and the curves become more peaked at \(\zeta = 0\). The maximum values of temperature predicted by the present model are smaller that those predicted by the model of Nielsen et al. (2008, 2010) and by Fialko and Khazan (2005). The latter gives wrong predictions for high speeds (see Figure A1c). On the contrary, for moderate speed all the three models are quite comparable (see Figure A1a).

We emphasize that this comparison assumes the time averages of the time variable quantities \(w_m, \tilde{\tau}\) and \(v\) over the temporal window of interest appearing in equation (14), so that comparison is indicative of a general behavior.

**A.2. Time evolution in the center of the melt layer**

Let us now consider the solution for the temperature evolution in the center of the melt
layer. From equation (15) we have that, the pre–factor $C \equiv \left( \sqrt{\frac{2\pi}{e}} + \pi \text{erf} \left( \frac{1}{\sqrt{2}} \right) - \sqrt{2\pi} \right)$ is nearly equal to 1.16. By considering that $w_m, \tau$ and $v$ are all positive quantities by definition, we have that $\tau^f$ can assume values greater than $T_m$, or, in other words, that we can have the superheating phenomenon. This feature has been also found in laboratory experiments by Nielsen et al. (2010) and it also confirmed by results reported in Figure A1.

A.3. Boundary condition at $\zeta = w_m^-(t)$

The Stefan problem (see equation (25)) relates the spatial derivatives of solid and melt temperature calculated at the melt–solid boundaries ($\zeta = \pm w_m(t)$) to the growth rate of the melt layer $w_m(t)$. Nielsen et al. (2008) found an approximate relation expressing heat flux $\tilde{q}_\zeta$ leading to equation (A1) previously reported.

From equation (14), by considering the time averages, we have:

$$\frac{\partial}{\partial \zeta} \bar{T}(\zeta, t) \bigg|_{\zeta = w_m^-(t)} \approx -\frac{1}{2} \tilde{c} \chi \text{erf} \left( \frac{1}{\sqrt{2}} \right) \bar{\tau} \bar{v}$$  \hspace{1cm} (A6)

which in agreement with (A1), taking into account that $\text{erf} \left( \frac{1}{\sqrt{2}} \right) = 0.7$. 
Appendix B. A posteriori verification that $\partial \bar{T} / \partial t$ term in equation (6) can be neglected

Equation (7) assumes that $\frac{\bar{q}}{c}$ is maximum for $\zeta = 0$; from equation (13) we have:

$$\frac{\bar{q}}{c} = \frac{\eta}{\sqrt{2\pi} c \varphi_m(t)} v^2(t)$$  \hspace{1cm} (B2)

In conclusion, equation (15) is a valid solution of (6) if the following condition is satisfied:

$$\frac{1}{v(t)} \left| \frac{\partial v(t)}{\partial t} \right| \ll \frac{\sqrt{2\pi} \varphi}{C} \frac{1}{\varphi_m(t)}$$  \hspace{1cm} (B3)

for all arbitrary times $t$ such that such that $\bar{\tau} = \tau^{(NS)}$. Interestingly, condition (B3) is
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independent of the dynamic viscosity; it only relates melt layer half-thickness for fault slip velocity and fault slip acceleration. Equation (B3) may be also physically interpreted as a condition on the melt layer thickness; until \( w_m(t) \) is such that

\[
\frac{w_m^2(t)}{\frac{\sqrt{2\pi} \, \bar{\chi}}{C}} \ll \left| \frac{v(t)}{\frac{\partial v(t)}{\partial t}} \right|
\]  \hspace{1cm} (B4)

the equation (15) is a valid solution of (6). On the contrary, when condition (B4) is violated the analytical solution (15) is no longer valid and therefore (6) can be solved only numerically.

We also conclude emphasizing that the condition \( \left| \frac{\partial \bar{T}}{\partial t} \right| \ll \left| \frac{q}{c} \right| \) does not conflict with the time variability of \( \bar{T} \), explicitly stated in equations (14) and (15) and obtained in our numerical experiments.

The numerical experiments presented and discussed in the present paper satisfy both the conditions (B3) and (B4).
Appendix C. Validity of equation (18) for dynamic viscosity evolution

The Arrhenius equation applied to dynamic viscosity of a melt material (see Shaw, 1972; Dingwell, 1998) postulates a dependence on the absolute temperature, which reads:

\[ \tilde{\eta} = \tilde{K} \, e^{\frac{\tilde{\tau}_a}{T}} \]  (C1)

where \( \tilde{K} \) is a constant pre–exponential factor and \( \tilde{\tau}_a \) is the activation temperature. Equation (18) follows from (C1) simply by putting the absolute temperature inside the melt layer (so that \( T \) in (C1) is expressed as \( \tilde{T} + 273.15 \), where \( \tilde{T} \) as in equation (14) or (15) is in °C). Moreover, in equation (18) we consider the temperature at the previous elasto–dynamic time step \( (t - \varepsilon) \) formally is \( t^{(m)} - \Delta t \), where \( t^{(m)} = m \Delta t \) is the discrete time at level \( m \) and \( \Delta t \) is the time step.

In this appendix we will quantitatively evaluate the goodness of such an assumption by considering typical scenarios for temperature evolution. In the synoptic comparison we will also consider the Nahme’s approximation of (C1), which has been often considered in the literature (Costa and Macedonio, 2002; Nielsen et al., 2008, 2010):

\[ \tilde{\eta} = \tilde{\eta}_m \, e^{\frac{\tilde{T} - \tilde{T}_m}{\tilde{T}_c}} \]  (C2)

where \( \tilde{\eta}_m = \tilde{K} \, e^{\frac{\tilde{\tau}_a}{\tilde{T}_c}} \) and \( \tilde{T}_c \equiv T_m^2 / \tilde{T}_a \). After simple algebra we can rewrite (C2) as follows:
By assuming the same parameters as Nielsen et al. (2008), \( \bar{\eta}_m = 1 \times 10^4 \) Pa s, \( \bar{T}_c = 75 ^\circ C = 348.15 \) K and \( T_m = 1200 ^\circ C = 1473.15 \) K, we obtain the two parameters of (C1) as listed in Table 1 (\( \bar{K} = 154.37 \) Pa s and \( \bar{T}_a = 6233 \) K).

In Figure C1 we report the comparison (by adopting the above–mentioned values) of the original Arrhenius law (red curve), its Nahme’s approximation (blue curve) and the approximation adopted in the present paper, equation (18), as obtained by assuming that the temperature increase from a time level to its subsequent time level is 5, 10, 20 and 50 K (black and gray curves). In other words we assume that \( \Delta T \equiv T(t_m) - T(t_m - \Delta t) = 5, 10, 20 \) and 50 K.

From Figure C1 we can immediately see that the Nahme’s approximation is valid only for a small interval after the melting point; this is not surprising, since equation (C2) has been obtained by considering the Taylor expansion of the term \( \bar{T}_a/T \) appearing in (C1) in the vicinity of \( T = T_m \). Nevertheless, Figure C1 clearly demonstrates that for temperatures greater than about 1600 °C (which can be easily realized in the center of the melt layer; see Figure 4b in Nielsen et al., 2008) equation (C2) significantly differs from (C1), more than the approximation used in the present paper. From Figure C1b we have that at \( T = 1600 \) °C the Nahme’s approximation differs from true value of the Arrhenius equation more than 21 % of the Arrhenius value. On the contrary, with the approximation used in the present study (equation (18)), these percentage differences are 0.9 %, 1.8 %, 3.7 % and 9.6 % for \( \Delta T = 5, 10, \)
20 and 50 K, respectively. In other words, equation (C2) gives a biased value of dynamic viscosity for temperature roughly greater than $T_m + 300$ °C. On the contrary, we can notice an overall good agreement of the approximation adopted in the present paper ($T = \bar{T} (t^{(m)} - \Delta t) + 273.15$) with respect to the reference case of equation (C1) for a wide range of temperatures.

For temperatures close to $T_m$ the agreement between (C1) and (18) remains good (see Figures C1a and C1b), provided that the increments of temperature from one elasto–dynamic time level and its subsequent are 20 K at maximum. This condition can be easily satisfied by considering a sufficiently fine temporal discretization, as such adopted in the numerical experiments presented in this paper (see Table 1).

The differences between (C1) and (18) for temperatures near the melting point can be easily reduced as follows. Let us consider in equation (18) the pre–exponential constant to be

$$
\tilde{K}' \equiv \tilde{K} e^{-\tilde{\tau}_s \Delta \bar{T} \left( \tilde{\tau}_s - \Delta \bar{T} \right)}
$$

instead of $\tilde{K}$. In such a way at the melting temperature $T_m$ all the approximations give exactly the same value of dynamic viscosity, i.e., $\tilde{\eta}_m = 1 \times 10^4$ Pa s. The replacement of $\tilde{K}$ with $\tilde{K}'$ merely represents a shift in the ordinate axis. (In the ideal case of arbitrarily small $\Delta \bar{T}$ we have that $\tilde{K}'$ of equation (C4) reduces to $\tilde{K}$).

The behavior of $\tilde{\eta}$ as predicted by the various approximation for increasing temperatures is reported in Figure C2a; in the present case we use in equation (19) $\tilde{K}'$ as in (C4). From Figure C2a we can immediately see that the agreement of our approximate relation (equation (18))
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and the original Arrhenius equation is now remarkably good over the whole range of considered temperatures (compare the inset panels in Figures C1b and C2b). At large temperatures the agreement remains good; at $T = 1600 \, ^\circ\text{C}$ the percentage misfits are now: – 0.6 %, – 1.1 %, – 2.2 % and – 5.6 % for $\Delta T = 5, 10, 20$ and 50 K, respectively (see Figure C2b).
References


Bizzarri, A., and M. Cocco (2006a), A thermal pressurization model for the spontaneous dynamic rupture propagation on a three-dimensional fault: 1. Methodological approach,


Holland, T., and R. Powell (1990), An enlarged and updated internally consistent thermodynamic dataset with uncertainties and correlations, *J. Metamorph. Geol.*, 8,


Otsuki, K., N. Monzawa, and T. Nagase (2003), Fluidization and melting of fault gouge during seismic slip: Identification in the Nojima fault zone and implications for focal earthquake


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Figure Captions

Figure 1. Sketch representing the considered fault structure. At a generic time after the onset of melting, a melt layer of thickness $2w_m(t)$ and enlarging with rate $\dot{w}_m(t)$ exists within a slipping zone $2w$ thick. The latter is surrounded by the damage zone. The plane $\zeta = 0$ defines, in the Cartesian reference system $O\xi_1\xi_2\xi_3$, the center of the slipping zone (i.e., the idealized fault plane). The coordinate $\xi$ is normal to the fault plane and it is anchored to the melt–solid boundary ($\xi \equiv \zeta - w_m(t)$).

Figure 2. Geometry of the model. The light grey plane indicates the fault $x_2 = x_2^f$, while the grey box marks the portion of the computational domain where the calculations are performed, due to the exploitation of the symmetry about the hypocenter H and about the fault plane (see Bizzarri, 2009 for details).

Figure 3. Comparison between results neglecting (left column) and considering (right column) melting and viscous shear. Top panels report the distribution of the rupture velocity ($v_r$) on the fault plane ($v_r$ is the inverse of rupture time gradient). Bottom panels report the maximum (peak) fault slip velocity. The model without melting effects assumes a linear SW friction law (equation (16)), while rheology of model with melting is described in details in sections 5.2 and 5.3. The adopted parameters are those tabulated in Table 1 and melt layer evolves as in equation (27).
Figure 4. Solutions corresponding to Figures 3b and 3d in a fault point located at the hypocentral depth and at a distance of 3 km from H. (a) Time evolution of frictional resistance. (b) Evolution of temperature change (referred to $T^f_0$ and calculated through equation (24)). (c) Phase portrait (i.e., traction vs. slip velocity). (d) Evolution of the dynamic viscosity (see equation (18)). In all panels the grey portions of the curves (after $t_{eff} = 0.93$ s) emphasize when the fault is governed by a viscous rheology (conversely, black portions indicate where SW friction law is paramount).

Figure 5. Slip–weakening curve corresponding to the solution reported in Figure 4. Circles represent data field observations from thrusts faults in Outer Hebrides, Scotland (see text for details). Red curve is the estimate of the fault traction as given by equation (28).

Figure 6. Effects of different time evolutions of the melt layer thickness in the case of a slipping zone 2 mm thick. $w_m$ follows equation (27) and the different values of $\dot{w}_m$ are reported in the legends (the other parameters are those of Table 1). (a) Time evolution of traction. (b) Time evolution of temperature change. (c) Slip–weakening curve. (d) Phase portrait. Vertical lines indicate when melting locally starts ($t_m$); big full circles eventually denote the point where the $w_m = w$ (end limit of our simulations). Black curves pertain to the reference simulation, where melting effects are not considered.

Figure 7. The same as in Figure 6, but now with $w = 7$ mm.
**Figure 8.** Distribution of temperature on the fault plane at $t = 1.68$ s for a slipping zone 2 mm thick and a melting zone growth rate of 0.1 mm/s (this corresponds to yellow curves in Figure 6).

**Figure 9.** (a) Distribution of melting instant on the fault plane, showing that the minimum is located in the hypocenter $H$. Purple region denotes the portion of the fault at rest. (b) Corresponding shape of the melt layer half–thickness as resulting from equation (27). The referential system $O_x^\xi \xi_1 \xi_3$ of Figure 1 is reported for clarity. Both panels refer to a numerical simulation where $w = 1$ mm and $\dot{w}_m = 0.1$ mm/s.

**Figure 10.** Profiles of $w_m$ as a function of depth calculated at 3 km from the hypocenter along the strike direction (as in Figures 4, 5, 6 and 7). Each line is computed every 0.1 s up to the last time level considered in the numerical experiments. Two representative growth rates are assumed; $\dot{w}_m = 0.1$ mm/s in panel (a) (as in Figure 9) and $\dot{w}_m = 1$ mm/s in panel (b). In both cases the slipping zone is 2 mm thick. In panel (b) dashed lines emphasize when the melt layer thickness exceeds $2w$ (upper limit in our model; see section 2 for further details). Values of melting instant (as defined by equation (1)) at $x_1 = 5$ km and $x_3 = x_3^H$ are also indicated.

**Figure A1.** Comparison between the solution obtained by Nielsen et al. (2008, 2010), $\tilde{T}^{NEA}(\zeta)$ (see equation (A2)), that from Fialko and Khazan (2005), $\tilde{T}^{NEA}(\zeta)$ (equation (B5)) and the time averaged solution obtained in the present paper, $\overline{T}(\zeta)$ (see equation 49).
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For the comparison we use \( \bar{\tau} = \tau_a = 20 \) MPa and \( \bar{\nu}_m = \nu_m^* \) as given by equation (A3); the other constitutive parameters are tabulated in Table 1. Values in ordinate axis are in °C. (a) \( \bar{\nu} = \nu_a = 0.1 \) m/s. (b) \( \bar{\nu} = \nu_a = 1 \) m/s. (c) \( \bar{\nu} = \nu_a = 10 \) m/s. The resulting values of \( \nu_m^* \) are 23.1 \( \mu \)m, 37.8 \( \mu \)m and 8.45 \( \mu \)m, respectively.

Figure C1. (a) Comparison between different evolution laws for dynamic viscosity of the melt material for a typical temperature interval above the melting point. Red curve is the original Arrhenius law (equation (C1)). Blue curve is its Nahme’s approximation (equation (C2), or equivalently (C3)). Black and grey curves pertain to the approximation adopted in the present paper, in which the temperature entering in the Arrhenius equation is computed at the previous time level (see equation (18)). (b) Normalized differences of the various approximations with respect to the true prediction given by the Arrhenius equation (C1); in the ordinate axis, for each temperature value, we plot the quantity

\[
100 \frac{\eta^{(\text{Approximation})} - \eta^{(\text{Arrhenius})}}{\eta^{(\text{Arrhenius})}}.
\]

In the legends are indicated the different values of the differences of temperature, \( \Delta T \) at time level, \( m \), and at its subsequent time level, \( m + 1 \) (\( \Delta T = 5, 10, 20 \) and 50 K). The adopted parameters are those listed in Table 1.

Figure C2. The same as in Figure C1, but now black and grey curves refer to our approximation (18) with the pre–exponential factor as in equation (C4); in such a way in all cases the dynamic viscosity at melting temperature identically equals \( \bar{\eta}_m = 1 \times 10^4 \) Pa s.

The values of \( \bar{K} \) from equation (C4) are 143.29, 141.23, 137.15 and 125.29 Pa s, for
$\Delta \bar{T} = 5, 10, 20$ and $50 \text{ K}$, respectively.
Table

Table 1. Parameters adopted in the present study; they refer to Gabbro.

<table>
<thead>
<tr>
<th><strong>Parameter</strong></th>
<th><strong>Value</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Medium and Discretization Parameters</strong></td>
<td></td>
</tr>
<tr>
<td>Lamé constants, $\lambda = G$</td>
<td>35.9 GPa</td>
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<tr>
<td>$S$ wave velocity, $v_S$</td>
<td>3.464 km/s</td>
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<tr>
<td>$P$ wave velocity, $v_P$</td>
<td>6 km/s</td>
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<tr>
<td>Fault length, $L_f$</td>
<td>16 km</td>
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<tr>
<td>Fault width, $W_f$</td>
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<tr>
<td>Spatial grid size, $\Delta x_1 = \Delta x_2 = \Delta x_3 \equiv \Delta x$</td>
<td>25 m</td>
</tr>
<tr>
<td>Time step, $\Delta t$</td>
<td>$6 \times 10^{-4}$ s $^{(a)}$</td>
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<tr>
<td><strong>Constitutive Parameters</strong></td>
<td></td>
</tr>
<tr>
<td>Effective normal stress, $\sigma_{\text{eff}}$</td>
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<tr>
<td>Initial shear stress (pre–stress), $\tau_0$</td>
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<tr>
<td>Upper yield stress, $\tau_u$</td>
<td>81.24 MPa</td>
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<tr>
<td>Kinetic friction level, $\tau_f$</td>
<td>55.2 MPa $^{(b)}$</td>
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<tr>
<td>Characteristic slip–weakening distance, $d_0$</td>
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<td><strong>Thermal Parameters</strong></td>
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<tr>
<td>Initial temperature in the center of the slipping zone, $T_0^f$</td>
<td>210 °C</td>
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<tr>
<td>Melting temperature, $T_m$</td>
<td>1200 °C $^{(c)}$</td>
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<tr>
<td>Latent heat of fusion, $L$</td>
<td>$350 \times 10^3$ J/kg</td>
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<tr>
<td>Slipping zone thickness (reference), $2w$</td>
<td>2 mm</td>
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<tr>
<td><strong>Solid state</strong></td>
<td><strong>Molten state</strong></td>
</tr>
<tr>
<td>Cubic mass density, $\rho$ or $\bar{\rho}$ $^{(d)}$</td>
<td>2990 kg/m$^3$</td>
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<tr>
<td>Heat capacity for unit volume of the composite, $c$ or $\bar{c}$ $^{(d)}$</td>
<td>$2.838 \times 10^6$ J/(m$^3$ °C)</td>
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<tr>
<td>Thermal diffusivity, $\chi$ or $\bar{\chi}$ $^{(d)}$</td>
<td>$0.344 \times 10^{-6}$ m$^2$/s</td>
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<tr>
<td>Arrhenious constant, $\tilde{K}$</td>
<td>n/a</td>
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<tr>
<td>Activation temperature, $\tilde{T}_a$</td>
<td>n/a</td>
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</table>
For the adopted parameters the Courant–Friedrichs–Lewy ratio, \( \omega_{CFL} = \frac{v_s \Delta t}{\Delta x} \), equals 0.083 and the estimate of the critical frequency for spatial grid dispersion, \( f_{acc}^{(c)} = \frac{v_s}{(6 \Delta x)} \), equals 23 Hz.

This results in a strength parameter \( S = 0.4 \left( \frac{\tau_u - \tau_0}{(\tau_0 - \tau_f)} \right) \).

As in Nielsen et al. (2008), we assume a single value of \( T_m \) even if each mineral composing the material assemblage in the slipping zone can have a different melting temperature, leading to martial melts and poly–phases; see Spray (1992).

Extrapolation from Holland and Powell (1990); for the solid state we assume an average temperature between \( T_0^f \) and \( T_m \).

With these values of \( \tilde{K} \) and \( \tilde{T}_a \) we obtain the dynamic viscosity at melting point

\[
\tilde{\eta}_m = \tilde{\eta} \left( \tilde{T} = T_m \right) = \tilde{K} \frac{T_f}{T_a + 273.15} = 1 \times 10^4 \text{ Pa s} \]
(as in Nielsen et al., 2008).
Viscous rheology

\[ \tau_f \]

Equation (28)

\[ d_0 \quad u(t_m^{\text{eff}}) \]

Traction (Pa)

Slip (m)
**Graphs and Data**

(a) Traction vs. Time

- **No melting**
- \( \dot{w}_m = 0.5 \text{ mm/s} \)
- \( \dot{w}_m = 5 \text{ mm/s} \)

**Total traction drops (MPa):**
- 26.04
- 72.60
- 79.95

(b) Temperature change vs. Time

- \( T_{max} = 12355 \text{ °C} \)
- \( T_{max} = 3653 \text{ °C} \)

(c) Traction vs. Slip

- **No melting**
- \( \dot{w}_m = 0.5 \text{ mm/s} \)
- \( \dot{w}_m = 5 \text{ mm/s} \)

(d) Traction vs. Slip velocity

- **No melting**
- \( \dot{w}_m = 0.5 \text{ mm/s} \)
- \( \dot{w}_m = 5 \text{ mm/s} \)
$w = 1 \text{ mm, } \dot{w}_m = 0.1 \text{ mm/s}$

$T = T^f_0 = 210 \, ^\circ\text{C}$
(a) $w = 1 \text{ mm}, \dot{w}_m = 0.1 \text{ mm/s}$

(b) $w_m = 0$

Along depth distance (m)

Along strike distance (m)

Melting instant (s)

Melt layer half-thickness (m)
(a) $w = 1\text{ mm}, \dot{w}_m = 0.1\text{ mm/s}, x_1^* = 5\text{ km}$

Profiles from $t = 1.0\text{ s}$ every $0.1\text{ s}$ ($t_m(x_1^*, x_3^H) = 0.98\text{ s}$)

$T^f < T_m$

$(b) w = 1\text{ mm}, \dot{w}_m = 1\text{ mm/s}, x_1^* = 5\text{ km}$

Profiles from $t = 0.9\text{ s}$ every $0.1\text{ s}$ ($t_m(x_1^*, x_3^H) = 0.83\text{ s}$)
Solution by Fialko and Khazan (2005) – Equation (A5)


Present solution – Equation (A4)