Multifractal Omori–Utsu Law for Earthquake Triggering
Theory and empirical tests
Single body approach
Spatial correlations between Coulomb stress and aftershocks but no account of the stress fluctuations due to aftershocks.
Single body approach

Earthquake nucleation activated by static stress

Spatial correlations between Coulomb stress and aftershocks but no account of the stress fluctuations due to aftershocks.
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State and rate friction
Dieterich (1994)

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Stress corrosion
Yamashita and Knopoff (1987)

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Single body approach

Time dependence

State and rate friction
Dieterich (1994)

Stress corrosion
Yamashita and Knopoff (1987)
Earthquake nucleation activated by static stress

Time Dependence

State and rate friction
Dieterich (1994)

Omori law with $p=1$
independent of $M$

Spatial correlations between Coulomb stress and aftershocks but no account of the stress fluctuations due to aftershocks.

Stein et al., 1994

Stress corrosion
Yamashita and Knopoff (1987)
The need of a manybody approach
The need of a manybody approach

Complex previous history with known or unknown rupture parameters

Stein et al., 1994
The need of a manybody approach

State and rate friction

Time dependence

Earthquake nucleation activated by static stress

Complex previous history with known or unknown rupture parameters

Stress corrosion

Stein et al., 1994
The need of a manybody approach

Earthquake nucleation activated by static stress

Complex previous history with known or unknown rupture parameters

State and rate friction

Stress corrosion

Omori law with $p=??$

Time dependence
The physical model: thermal activation driven by stress
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Energy barrier = \( E_0 - E(t) \)
The physical model: thermal activation driven by stress

Before the shock

Energy barrier = $E_0 - E(t)$

Arrhenius law for the activation rate:

$$\lambda(t) = \lambda_0 \exp\left(-\frac{E_0 - E(t)}{kT}\right)$$

After the shock
The physical model: thermal activation driven by stress

Arrhenius law for the activation rate:

\[ \lambda(t) = \lambda_0 \exp \left( - \frac{E_0 - E(t)}{kT} \right) \]

Energy barrier = \( E_0 - E(t) \)

stress barrier = \( \sigma_0 - \sigma(t) \)
The physical model: thermal activation driven by stress

Before the shock

Energy barrier = \( E_0 - E(t) \)

After the shock

\[ \lambda(t) = \lambda_0 \exp\left( -\frac{E_0 - E(t)}{kT} \right) \]

Arrhenius law for the activation rate:

stress barrier = \( \sigma_0 - \sigma(t) \)

\[ \lambda(t) = \lambda_0 \exp\left( -\frac{\sigma_0 - \sigma(t)}{kT} V \right) \]

\( \lambda(t) \): instantaneous rate
\( \lambda_0 \sim \) average nucleation rate
\( \sigma_0 \): material strength
\( \sigma(t) \): applied stress
\( V \): activation volume
\( T \): temperature
\( k \): Boltzmann constant

Compatible with state-and-rate friction, stress corrosion, ...

...
Taking account of history and boundary conditions

\[ \lambda_0' = \lambda_0 \exp\left(-\frac{\sigma_0 V}{kT}\right) \]
Taking account of history and boundary conditions

\[ \lambda_0' = \lambda_0 \exp \left( -\frac{\sigma_0}{kT} V \right) \]

\[ \lambda (r, t) = \lambda_0' \exp \left( \frac{\sigma(r, t)}{kT} V \right) \]

Stress is assumed to be a scalar for the sake of simplicity
Taking account of history and boundary conditions

\[ \lambda' = \lambda_0 \exp\left(-\frac{\sigma_0}{kT} V\right) \]

\[ \lambda(r, t) = \lambda'_0 \exp\left(\frac{\sigma(r, t)}{kT} V\right) \]

Stress is assumed to be a scalar for the sake of simplicity

\[ \sigma(r, t) = \sigma(r)_{\text{far field}} + \int_{-\infty}^{t} \int dN \left[ d\rho \times d\tau \right] \Delta \sigma(\rho, \tau) \cdot G(r - \rho, t - \tau) \]

local stress
Taking account of history and boundary conditions

\[ \lambda' = \lambda_0 \exp \left( -\frac{\sigma_0}{kT} V \right) \]

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- local stress
- tectonic loading
Taking account of history and boundary conditions

\[ \lambda' = \lambda_0 \exp\left(-\frac{\sigma_0 V}{kT}\right) \]

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- **local stress**
- **tectonic loading**

Stress fluctuations induced by all past events in the system
Taking account of history and boundary conditions

\[ \lambda_0' = \lambda_0 \exp\left( -\frac{\sigma_0}{kT} V \right) \]

\[ \lambda(r, t) = \lambda_0' \exp\left( \frac{\sigma(r, t)}{kT} V \right) \]

Stress is assumed to be a scalar for the sake of simplicity

\[ \sigma(r, t) = \sigma(r)_{\text{far field}} + \int_{-\infty}^{t} \int_{\text{space}} dN [d\rho \times d\tau] \Delta \sigma(\rho, \tau) G(r - \rho, t - \tau) \]

local stress tectonic loading
Taking account of history and boundary conditions

\[ \lambda' = \lambda_0 \exp\left(-\frac{\sigma_0}{kT}V\right) \]

\[ \lambda(r, t) = \lambda'_0 \exp\left(\frac{\sigma(r, t)}{kT}V\right) \]

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local stress  tectonic loading
Taking account of history and boundary conditions

\[ \lambda_0' = \lambda_0 \exp \left( -\frac{\sigma_0}{kT} V \right) \]

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- Local stress
- Tectonic loading
- Time and space distribution of past sources
Taking account of history and boundary conditions

\[ \lambda' = \lambda_0 \exp\left( -\frac{\sigma_0}{kT} \right) \]

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- local stress
- tectonic loading
- Time and space distribution of past sources
- Stress fluctuations at sources
Taking account of history and boundary conditions

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\[ \sigma(r, t) = \sigma(r)_{\text{far field}} + \int_{-\infty}^{t} \int dN \left[ d\rho \times d\tau \right] \Delta \sigma(\rho, \tau) G\left( r - \rho, t - \tau \right) \]

- local stress
- tectonic loading
- Time and space distribution of past shocks
- Stress fluctuations at sources
- Green function for stress transfer
A few working hypotheses
A few working hypotheses

Every shock is activated by stress and temperature according to Arrhenius law.
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Every shock of magnitude $M$ triggers instantaneously $10^{qM}$ aftershocks (ETAS–like productivity law)
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Every shock of magnitude $M$ triggers instantaneously $10^{qM}$ aftershocks (ETAS–like productivity law)

Separation of variables:

$$G(r,t) = f(r) \times h(t)$$
A few working hypotheses

Every shock is activated by stress and temperature according to Arrhenius law.

Every shock of magnitude $M$ triggers instantaneously $10^{qM}$ other events.

Separation of variables: $G(r,t) = f(r) \times h(t)$

Stress fluctuations depend on the location of events (red dots), their rupture geometry, and on the spatial decay of the Green function. Most of these parameters are unknown, and some events even not recorded at all. Those fluctuations are thus considered as realizations of a random variable.
A few working hypotheses

Every shock is activated by stress and temperature according to Arrhenius law

Every shock of magnitude $M$ triggers instantaneously $10^{qM}$ other events

Separation of variables:

$$G(r, t) = f(r) \times h(t)$$

Stress fluctuations at location due to previous events:

$$P(\sigma_{\text{fluc}}) d\sigma_{\text{fluc}} \approx \frac{C}{(\sigma_{\text{fluc}} + \sigma_{f1})^{\mu}} d\sigma_{\text{fluc}}$$

Exponent $\mu$ depends on (and encapsulates) the spatial structure of the fault pattern, the GR law, as well as $f(r)$. 
A few working hypotheses

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Exponent $\mu$ depends on (and encapsulates) the spatial structure of the fault pattern, the GR law, as well as $f(r)$.

Elastoviscoplastic rheology

$$h(t) = \frac{h_0}{(t + t_1)^{\mu}} \exp\left(-\frac{t}{\tau_M}\right)$$

Maxwell time

$\tau_M \gg$ time scale of observations

$h(t)$: dislocations motion and unresolved seismicity
Relaxation after a magnitude M event
Relaxation after a magnitude $M$ event

We re–write in discrete form after spatial averaging:

$$\lambda(t) = \lambda_{tec} \exp \left[ \frac{V}{kT} \sum_{past} \sigma_{fluc}(t_i) h(t - t_i) \right]$$

cf Ouillon and Sornette, JGR, 2005

$\lambda_{tec}$ is the average seismicity rate, modulated by a time–varying activation term. The formulation is thus non–linear.
Relaxation after a magnitude M event

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\[ P(\sigma_{fluc}) d\sigma_{fluc} = \frac{C}{(\sigma_{fluc} + \sigma_1)^\mu} d\sigma_{fluc} \]

\[ h(t) = \frac{h_0}{(t + t_1)^\theta} \exp \left( -\frac{t}{\tau_M} \right) \]
Relaxation after a magnitude M event

We re-write in discrete form after spatial averaging:

$$\lambda(t) = \lambda_{tec} \exp \left[ \frac{V}{kT \sum_{past} \sigma_{fluc}(t_i)h(t - t_i)} \right]$$

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$$\lambda_{tec}$$ is the average seismicity rate, modulated by a time-varying activation term. The formulation is thus non-linear.

$$\lambda(t) \propto t^{-p(M)}$$

$$p(M) = aM + b$$

Power-law relaxation rate of aftershocks increases with the size of energy fluctuations => multifractality

Multifractal Stress Activation model
Probable meaning of multifractality
Probable meaning of multifractality

\[ P(\sigma_{fluc}) d\sigma_{fluc} \approx \frac{C}{(\sigma_{fluc} + \sigma_f)^{\mu}} d\sigma_{fluc} \]

\( \mu \): Spatial self-organization of the fault pattern

Fault network geometry  \rightarrow  Seismicity
Probable meaning of multifractality

\[ P(\sigma_{\text{fluc}}) d\sigma_{\text{fluc}} \approx \frac{C}{(\sigma_{\text{fluc}} + \sigma_f)^\mu} d\sigma_{\text{fluc}} \]

\[ h(t) = \frac{h_0}{(t + t_1)^{\mu+\theta}} \exp\left(-\frac{t}{\tau_M}\right) \]

\[ \theta : \text{temporal self-organization of the fault pattern} \]
Probable meaning of multifractality

\[ \mu(1+\theta) = 1 : \]

Spatio-temporal self-organization of the fault pattern

\[ h(t) = \frac{h_0}{(t + t_1)^{\mu + \theta}} \exp \left( -\frac{t}{\tau_M} \right) \]

Fault network geometry

seismicity

seismicity

Stress relaxation
Building aftershocks time series
Select all events within a given magnitude range \([M_1;M_2]\).
Building aftershocks time series

Select all events within a given magnitude range $[M_1;M_2]$. 

Define a spatial window ($R=2L$)
Building aftershocks time series

Select all events within a given magnitude range $[M_1;M_2]$.

Define a spatial window $(R=2L)$

Define a time window $(T=1 \text{ year})$
Building aftershocks time series

Select all events within a given magnitude range \([M_1;M_2]\).

Define a spatial window \((R=2L)\)

Define a time window \((T=1\text{ year})\)

Consider all events within \([0,R] \times [0,T]\) as aftershocks
Define a time window ($T=1$ year)

Define a spatial window ($R=2L$)

Consider all events within $[0,R] \times [0,T]$ as aftershocks.

If the starting event is the aftershock of a larger event, remove it and its aftershocks.
Building aftershocks time series

Select all events within a given magnitude range \([M_1;M_2]\).

Define a spatial window \((R=2L)\)

Define a time window \((T=1 \text{ year})\)

Consider all events within \([0,R] \times [0,T]\) as triggered events

If the starting event is the aftershock of a larger event, remove it and its aftershocks

Stack all individual aftershocks series
Building aftershocks time series

Select all events within a given magnitude range \([M_1;M_2]\).

Define a spatial window \((R=2L)\)

Define a time window \((T=1\text{ year})\)

Consider all events within \([0,R] \times [0,T]\) as triggered events

If the starting event is the aftershock of a larger event, remove it and its aftershocks

Stack all individual aftershocks series

\[ N(t) = A \times t^{-p} + B(t) \]
Define a time window ($T=1$ year)

Define a spatial window ($R=2L$)

Consider all events within $[0,R] \times [0,T]$ as triggered events

Select all events within a given magnitude range $[M_1; M_2]$.

If the starting event is the aftershock of a larger event, remove it and its aftershocks

Stack all individual aftershocks series

Time distribution of aftershocks

$$N(t) = A \times t^{-p} + B(t)$$

Omori law
Define a time window ($T=1$ year)

Define a spatial window ($R=2L$)

Consider all events within $[0,R] \times [0,T]$ as triggered events

Select all events within a given magnitude range $[M_1;M_2]$.

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Stack all individual aftershocks series

Building aftershocks time series

\[ N(t) = A \times t^{-p} + B(t) \]
The Scaling Function Analysis (SFA)
The Scaling Function Analysis (SFA)

The background term is regular:

\[ B(t) = b_0 + b_1 t + b_2 t^2 + \ldots + b_n t^n \]
The Scaling Function Analysis (SFA)

The background term is regular

\[ B(t) = b_0 + b_1 \times t + b_2 \times t^2 + \ldots + b_n \times t^n \]

Scaling Function

\[ \Psi\left(\frac{t}{s}\right) = a_0 + a_1 \times \left(\frac{t}{s}\right) + a_2 \times \left(\frac{t}{s}\right)^2 + \ldots + a_m \times \left(\frac{t}{s}\right)^m \times \exp\left(-5 \left(\frac{t}{s}\right)^2\right) \]

Time scale \( s \) is chosen by the user.
The Scaling Function Analysis (SFA)

The background term is regular

\[ B(t) = b_0 + b_1 \times t + b_2 \times t^2 + \ldots + b_n \times t^n \]

Scaling Function

\[
\Psi\left(\frac{t}{s}\right) = a_0 + a_1 \times \left(\frac{t}{s}\right) + a_2 \times \left(\frac{t}{s}\right)^2 + \ldots + a_m \times \left(\frac{t}{s}\right)^m \times \exp\left(-5\left(\frac{t}{s}\right)^{\frac{2}{3}}\right)
\]

Time scale \( s \) is chosen by the user

All \( a_i \)'s and \( m \) are chosen such that

\[
\int_{0}^{\infty} \Psi\left(\frac{t}{s}\right) B(t) dt = 0
\]

(depending only on \( n \))
The Scaling Function Analysis (SFA)

The background term is regular:

\[ B(t) = b_0 + b_1 \, \tau + b_2 \, \tau^2 + \ldots + b_n \, \tau^n \]

The Scaling Function

\[
\Psi \left( \frac{t}{s} \right) = \left( a_0 + a_1 \left( \frac{t}{s} \right) + a_2 \left( \frac{t}{s} \right)^2 + \ldots + a_m \left( \frac{t}{s} \right)^m \right) \times \exp \left( -5 \left( \frac{t}{s} \right)^2 \right)
\]

Time scale \( s \) is chosen by the user.

All \( a_i \)'s and \( m \) are chosen such that

\[
\int_0^\infty \Psi \left( \frac{t}{s} \right) B(t) \, dt = 0
\]

(assuming only on \( n \))
The Scaling Function Analysis (SFA)

\[ N(t) = A \times t^{-p} + B(t) \]
The Scaling Function Analysis (SFA)

\[ N(t) = A \times t^{-p} + B(t) \]

Scaling Function Analysis Coefficient

\[ C(s) = \int_0^\infty \Psi \left( \frac{t}{s} \right) N(t) dt = s^{1-p} A \int_0^\infty \Psi(t) t^{-p} dt + \int_0^\infty \Psi \left( \frac{t}{s} \right) B(t) dt \]
The Scaling Function Analysis (SFA)

Scaling Function Analysis Coefficient

\[ C(s) = \int_0^\infty \Psi\left(\frac{t}{s}\right) N(t) dt = s^{1-p} A \int_0^\infty \Psi(t) t^{-p} dt + \int_0^\infty \Psi\left(\frac{t}{s}\right) B(t) dt \]

\[ N(t) = A t^{-p} + B(t) \]
The Scaling Function Analysis (SFA)

$N(t) = A \times t^{-p} + B(t)$

Scaling Function Analysis Coefficient

$C(s) = \int_0^\infty \Psi\left(\frac{t}{s}\right)N(t)dt = s^{1-p} \int_0^\infty \Psi(t) t^{-p} dt + \int_0^\infty \Psi\left(\frac{t}{s}\right)B(t)dt$
The Scaling Function Analysis (SFA)

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The Scaling Function Analysis (SFA)

Scaling Function Analysis Coefficient

\[ N(t) = A \times t^{-p} + B(t) \]

\[
C(s) = \int_0^\infty \Psi \left( \frac{t}{s} \right) N(t) dt = s^{1-p} \int_0^\infty \Psi(t) \times t^{-p} dt + \int_0^\infty \Psi \left( \frac{t}{s} \right) B(t) dt
\]
The Scaling Function Analysis (SFA)

Scaling Function Analysis Coefficient

\[ N(t) = A \cdot t^{-p} + B(t) \]

\[ C(s) = \int_0^{\infty} \Psi\left(\frac{t}{s}\right) N(t) dt = s^{1-p} \int_0^{\infty} \Psi(t) \cdot t^{-p} dt + \int \Psi\left(\frac{t}{s}\right) N(t) dt \]
The Scaling Function Analysis (SFA)

Scaling Function Analysis Coefficient

\[ C(s) = \int_0^\infty \Psi \left( \frac{t}{s} \right) N(t) dt = s^{1-p} A \int_0^\infty \Psi(t) t^{-p} dt + \int_0^\infty \Psi(t) B(t) dt \]

\[ C(s) \propto s^{1-p} \]
An example on a real catalog
An example on a real catalog

Ouillon et al, 2007 submitted to GJI

3 different scaling functions yield power law scaling with the same value of $p$. 

$$C(s)$$

$SCEC - M[2.5;3.0] - p=0.63$

$n=0$ $m=2$

$n=3$ $m=5$

$n=0$ $m=12$
Results on real catalogs
Results on real catalogs

\[ M = \frac{M_1 + M_2}{2} \]
Results on real catalogs

Construction of standard bined time series and least squares fits.

\[ N(t) = A \times t^{-p} + b_0 \]
Results on real catalogs

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Construction of standard binned time series and least squares fits.

\[ N(t) = A \times t^{-p} + b_0 \]
Results on real catalogs

SFA: $p(M) = 0.11M + 0.38$
Bins: $p(M) = 0.10M + 0.40$

SCEC 1932–200
Results on real catalogs

SFA: \( p(M) = 0.16M - 0.09 \)

Bins: \( p(M) = 0.13M + 0.14 \)


SCEC 1932–2006

SFA: \( p(M) = 0.11M + 0.38 \)

Bins: \( p(M) = 0.10M + 0.40 \)

SFA: \( p(M) = 0.16M - 0.09 \)

Bins: \( p(M) = 0.13M + 0.14 \)
Summary
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Mechanical model taking account of interactions between all events
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Seismicity rate depends exponentially on applied stress
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Stress fluctuations are distributed as power laws ($\mu$)
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Seismicity rate depends exponentially on applied stress

Stress fluctuations are distributed as power laws ($\mu$)

Stress fluctuations decay with time as power laws ($\theta$)

$\mu(1+\theta)=1 \Rightarrow p(M)=aM+b$ in agreement with empirical observations
Summary

Mechanical model taking account of interactions between all events

Seismicity rate depends exponentially on applied stress

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Stress fluctuations decay with time as power laws ($\theta$)

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This model is the only one that is able to predict the multifractal nature of seismicity
Summary

Mechanical model taking account of interactions between all events

Seismicity rate depends exponentially on applied stress

Stress fluctuations are distributed as power laws ($\mu$)

Stress fluctuations decay with time as power laws ($\theta$)

$\mu(1+\theta)=1 \Rightarrow p(M)=aM+b$ in agreement with empirical observations

This model is the only one that is able to predict the multifractal nature of seismicity

Multifractality stems from the spatio-temporal self-organization of the fault pattern ($\mu(1+\theta)=1$)