A quantitative model for volcanic hazard assessment

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Volcanic hazard assessment is a basic ingredient for risk-based decision-making in land-use planning and emergency management. Volcanic hazard is defined as the probability of any particular area being affected by a destructive volcanic event within a given period of time (Fournier d'Albe 1979). The probabilistic nature of such an important issue derives from the fact that volcanic activity is a complex process, characterized by several and usually unknown degrees of freedom that are often linked by nonlinear relationships (e.g. Bak *et al.* 1988). Except in sporadic cases, the result of this complexity is an intrinsic, and perhaps unavoidable, unpredictability of the time evolution of the volcanic system from a deterministic point of view.

In reality, current volcanic hazard assessment is even more entangled by scarce data and relatively poor knowledge of the physical processes. Cumulatively, these difficulties prevent a solution of the hazard or risk problem from a rigorous scientific perspective. Nevertheless, the potential for extreme risk pushes us to be pragmatic and to attempt to solve the problem from an 'engineering' point of view. Because of the devastating potential of volcanoes close to urbanized areas, the scientific community must address the issue as accurately and precisely as possible with the currently available methods and based on our current understanding of volcanic systems. This assessment is best done by treating scientific uncertainty in a fully structured manner. In this respect, Bayesian statistics provides a suitable framework for producing a volcanic hazard or risk assessments in a rational, probabilistic form (e.g. UNESCO 1972; Gelman *et al.* 1995). To illustrate the general philosophy of the approach, we quote Toffler (1990) who stated that 'it is better to have a general and incomplete map, subject to revision and correction, than to have no map at all'. In other words, the risk (hazard) assessment process implies that a limited database and knowledge is no excuse for not conducting sound hazard and risk assessment. On the contrary, with less knowledge of a system, the need for assessment and management of hazards and risk becomes more imperative (Haimes 2004).

In this paper we present and further develop the method proposed by Marzocchi *et al.* (2004) based on the event tree (Newhall & Hoblitt 2002) scheme to estimate the probability of all the relevant possible outcomes of a volcanic crisis and, in general, to quantify volcanic hazard and risk. Marzocchi *et al.* (2004) emphasized the volcanological aspects of hazard assessment, dividing the problem into three consequential steps that encompass (1) the logical sequence of acquisition of information, (2) use of past data to assess long-term volcanic hazard (from years to decades), and (3) use of monitoring observations to assess mid- to short-term volcanic hazard (from hours to a few years). Here, we provide a more formal and generalized description of the use of this information in a Bayesian statistical framework to assess long-term volcanic hazard. In particular, we describe the use of data and specific probability density functions in the prior and posterior statistical distributions to calculate transition probabilities on the event tree. Overall, this Bayesian approach provides a robust treatment of uncertainty that is crucial in the estimation of hazards and risks using event trees.

There are three important points about the general approach. First, the scheme can take all available information into account, from theoretical models to past data and monitoring measurements. Second, the use of these different types of data in a Bayesian framework provides a mechanism for continuously updating probabilities, and therefore both the long- and mid- to short-term volcanic hazard may be continuously revised if necessary. For example, long-term volcanic hazard assessments are often used to compare different kinds of hazards (e.g. volcanic, seismic, industrial, floods) that may affect the same area. Results of long-term hazard assessments are very useful for cost–benefit analysis of risk mitigation actions, and for appropriate land-use planning, such as location of settlements. As data and models related to hazards are continually changing,

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and risks may change rapidly with population growth, easy update of event trees is essential. In contrast, mid- to short-term hazard assessment in a Bayesian framework assists with actions for immediate vulnerability (and risk) reduction, for instance through evacuation of people from dangerous areas (Fournier d'Albe 1979). The third point is more technical, and probably the most innovative one from a volcanological perspective. With our scheme, giving a prior distribution to the probability of risky events, we deal in a formal way with two different types of uncertainty: aleatoric and epistemic.

The concept of aleatoric (stochastic) and epistemic (data- or knowledge-limited) uncertainties is of primary importance in hazard and risk studies (see, e.g. Woo 1999). Aleatoric uncertainty arises from the impossibility of predicting deterministically the evolution of a system because of its intrinsic complexity. Epistemic uncertainty is associated with limitations in our knowledge of the system. In general, aleatoric uncertainty produces an irreducible stochasticity (randomness) in outcomes, regardless of our physical knowledge of the system. In contrast, epistemic uncertainty is, in principle, reducible by increasing the number or quality of data and/or improving our knowledge of the physical system.

The remainder of this paper is divided into two parts. The first part describes the mechanics of implementing Bayesian methods in the event tree scheme. This is intended to provide a statistical framework for application in volcanic hazard assessment. The second part briefly provides a pragmatic example related to a hypothetical event tree, with the goal of giving the reader a sense of how to apply these methods at specific volcanoes to assess long-term volcanic hazard.

The event tree scheme

The event tree is a branching graph representation of events in which individual branches are alternative steps from a general prior event, state or condition, and that evolve through time into increasingly specific subsequent events. Eventually the branches terminate in final outcomes representing specific hazards or risks that may transpire in the future. In this way, an event tree attempts to graphically display all relevant possible outcomes of volcanic unrest in progressively higher levels of detail. Points on the graph at which new branches are created are referred to as nodes.

As this definition is mainly driven by the practical utility of the event tree, the branches at each node point to the whole set of different possible events, regardless of their probabilistic features. In other words, the events at each node need not be mutually exclusive or exhaustive. Here, however, we consider only the first nodes of a generic event tree (see Fig. 1), where the events at each node are mutually exclusive and exhaustive. In this case, definitions are constructed so that no sequence of events can proceed along more than a single branch of the event tree. This property makes the event tree comparable with the event trees usually reported in statistical literature (Smith 1988). In Fig. 1 we show the first four nodes of a general event tree for an explosive volcano (see Marzocchi *et al.* 2004): Node 1, unrest occurs within a given time interval (τ); Node 2, the unrest is due to magma, given that unrest is detected; Node 3, the magma will reach the surface (i.e. it will erupt) in a given time interval (τ), provided that the unrest has a magmatic origin; Node 4, the eruption will be of a certain magnitude (characterized by the volcanic explosivity index; VEI), given that there is an eruption.

Here, we describe a statistically meaningful approach to the estimation of the probability density function (pdf) of the probability of events at each node of the event tree, taking into account all of the information available. We consider only the long-term volcanic hazard that is assessed during a quiet period of the volcano. In this case, monitoring data are used only to watch for unrest, and we estimate a posterior pdf by using data related to past episodes of volcanic activity. When unrest is detected, we may revise the event tree to assess mid- to short-term volcanic hazards (see Marzocchi *et al.* 2004). Information used for long-term volcanic hazards may not be particularly useful. Instead, monitoring data becomes more important. Because these pdfs are determined, their combination allows us to evaluate the probability of any desired event (e.g. Aspinall *et al.* 2003; Marzocchi *et al.* 2004). As described in the following, use of these pdfs allows aleatory and epistemic uncertainties to be estimated.

Posterior density function at the nodes for the long-term volcanic hazard

Here, the objective of a Bayesian analysis is to estimate the posterior pdf at the nodes, through the Bayes rule (Bernardo & Smith 2000). That is, the Bayes rule is used to update the *a priori* belief about the probability at each node by including available past data. The evaluation of the long-term volcanic hazard is based on these posterior distributions.

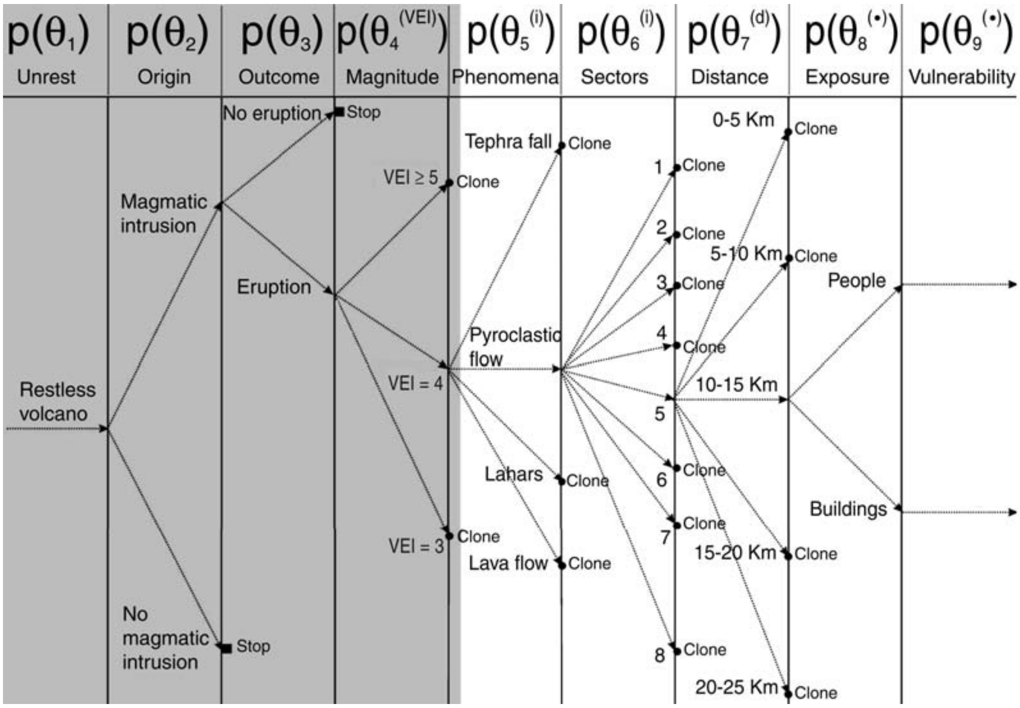

Fig. 1. Sketch of the event tree for Mt. Vesuvius as reported by Marzocchi *et al.* (2004). The nine steps of estimation progress from general to more specific events (left to right). It should be noted that any branch that terminates with 'Clone' is identical to the central branch. For example, in the column 'Sectors', the clones relative to sectors 1, 2, 3, 4, 6, 7 and 8 are identical to the central branch relative to sector 5. The branches terminating with 'Stop' are not discussed here because they cannot develop into dangerous subsequent events. The shaded part of the event tree is the part considered here.

At this point, we specify the data and the parameters used to calculate transition probabilities in the event tree node by node.

Node 1

The first node has two possible outcomes: 'occurrence of unrest' and 'not occurrence of unrest' in a given time window τ. These outcomes can be treated as 'success' and 'failure', respectively, in a binomial model.

(1) Let Y_1 be the variable that counts the number of (non-overlapping) time windows containing onset of unrest that occur in a set of n_1 inspections, the total number of non-overlapping time windows investigated (to prevent potential effects of possible past unrest that lasted a long time, n_1 considers only time windows that begin in a state of no unrest);

$$Y_1|\Theta_1 \sim \mathrm{Bin}(n_1, \theta_1) \qquad (1)$$

where θ_1 is the probability of unrest in the given time window τ, and Bin signifies a binomial pdf. The binomial density function is

$$f_{Y_1|\Theta_1}(y_1|\theta_1) = \binom{n_1}{y_1}\theta_1^{y_1}(1-\theta_1)^{n_1-y_1}. \qquad (2)$$

(2) We define the following prior distribution for Θ_1

$$\Theta_1 \sim \mathrm{Be}(\alpha_1, \beta_1) \qquad (3)$$

that is, as a Beta distribution with parameters α_1 and β_1, whose density function is

$$f_{\Theta_1}(\theta_1) = \frac{1}{B(\alpha,\beta)}\theta_1^{\alpha-1}(1-\theta_1)^{\beta-1},$$
$$0 < \theta_1 < 1, \alpha > 0, \beta > 0 \qquad (4)$$

where $B(\alpha, \beta)$ denotes the Beta function

$$B(\alpha,\beta) = \int_0^1 \theta_1^{\alpha-1}(1-\theta_1)^{\beta-1}\mathrm{d}x. \qquad (5)$$

The expected value for this Beta distribution is

$$E(\Theta_1) = \frac{\alpha_1}{\alpha_1 + \beta_1} \tag{6}$$

and the variance is

$$V(\Theta_1) = \frac{E(\Theta_1)(1 - E(\Theta_1))}{(\alpha_1 + \beta_1 + 1)}. \tag{7}$$

We chose the Beta distribution because it is defined in the range [0,1], and it is the conjugate prior distribution in the binomial model (Gelman *et al.* 1995). This choice is subjective and other distributions can be used, such as the Gauss distribution for the logistic transformation of the probability. In practice, the differences associated with the use of reasonable distributions are not usually significant; for this reason, the Beta distribution is the most used in practical applications (see, e.g. Gelman *et al.* 1995). If the present state of the art does not allow us to estimate a reasonable *a priori* model, we can express the state of maximum ignorance using the uniform distribution that corresponds to a Beta distribution with $\alpha_1 = \beta_1 = 1$.

(3) Through Bayes theorem and adopting the results of the conjugate families for the binomial model, we obtain the following posterior distribution for Θ_1:

$$\Theta_1 | Y_1 \sim \mathrm{Be}(\alpha_1 + y_1, \beta_1 + n_1 - y_1) \tag{8}$$

where y_1 is the number of time windows in which unrest is observed in the n_1 inspections.

Node 2

The structure of the second node is similar to the structure of the first node: success is 'magmatic intrusion' and failure is 'not magmatic intrusion', given that unrest has occurred.

(1) Let Y_2 be the variable that counts the number of magmatic intrusions that happen in a set of y_1 episodes of unrest that occurred at the previous node. Then we can write

$$Y_2 | Y_1, \Theta_2 \sim \mathrm{Bin}(y_1, \theta_2) \tag{9}$$

where θ_2 is the conditional probability of magmatic intrusion.

(2) We define the following prior distribution for Θ_2

$$\Theta_2 \sim \mathrm{Be}(\alpha_2, \beta_2). \tag{10}$$

Also in this case, the present state of the art suggests we set $\alpha_2 = \beta_2 = 1$.

(3) The posterior distribution for Θ_2 is

$$\Theta_2 | Y_2, Y_1 \sim \mathrm{Be}(\alpha_2 + y_2, \beta_2 + y_1 - y_2) \tag{11}$$

where y_2 is the number of magmatic intrusions observed in the set of y_1 periods of unrest from the previous node.

Node 3

The third node defines the 'occurrence of eruption' and 'not occurrence of eruption' in a given time window τ as the success and the failure in a binomial model, provided the unrest has magmatic origin.

(1) Let Y_3 be the variable that counts the number of eruptions that happen in a given time window τ in a set of y_2 magmatic intrusions that occurred at the previous node. Then

$$Y_3 | Y_2, \Theta_3 \sim \mathrm{Bin}(y_2, \theta_3) \tag{12}$$

where θ_3 is the conditional probability of eruption.

(2) We define the following prior distribution for Θ_3:

$$\Theta_3 \sim \mathrm{Be}(\alpha_3, \beta_3). \tag{13}$$

As before, our present state of knowledge of this process suggests we set $\alpha_3 = \beta_3 = 1$.

(3) Again we obtain the following posterior distribution for Θ_3:

$$\Theta_3 | Y_3, Y_2 \sim \mathrm{Be}(\alpha_3 + y_3, \beta_3 + y_2 - y_3) \tag{14}$$

where y_3 is the number of eruptions observed in the set of y_2 magmatic intrusions from the previous node.

Node 4

The fourth node represents the magnitude of the eruption in terms of VEI, categorized here as three possible outcomes j, where $j = 3, 4, 5+$ (VEI $= 3$, VEI $= 4$, VEI ≥ 5, respectively). It should be noted that we are considering a specific case where eruptions with VEI ≤ 2 are very unlikely, as for most explosive volcanoes in a closed conduit regime with little degassing. In this situation it is appropriate to assume that the next event must have at least the minimum energy required to reopen the conduit. However, generalizing this step to include lower VEI does not pose any conceptual difficulty. Let Y_4^j be the number of times that the event VEI $= j$ occurs in the set of y_3 eruptions observed in the previous node and let $Y_4 = (Y_4^3, Y_4^4, Y_4^{5+})$ be the vector of all the possible outcomes, where $Y_4^{5+} = y_3 - Y_4^3 - Y_4^4$.

(1) As a natural generalization of the binomial distribution, we hypothesize for Y_4 a multinomial distribution with three possible outcomes:

$$Y_4|Y_3,\ \Theta_4 \sim \mathrm{Mu}_3(y_3,\ \theta_4) \tag{15}$$

where $\theta_4 = (\theta_4^3, \theta_4^4, \theta_4^{5+})$ is the vector of the conditional probability for the three outcomes $\mathrm{VEI}=3$, $\mathrm{VEI}=4$, $\mathrm{VEI}\geq 5$, respectively (with $\theta_4^{5+} = 1 - \theta_4^3 - \theta_4^4$).

(2) We define the following prior distribution for Θ_4:

$$\Theta_4 \sim \mathrm{Di}_3(\alpha_4) \tag{16}$$

that is, a Dirichlet distribution with parameters $\alpha_4 = (\alpha_4^3, \alpha_4^4, \alpha_4^{5+})$, the conjugate prior distribution in the multinomial model. Such a distribution is the multivariate generalization of the Beta distribution (see Gelman *et al.* 1995), therefore the rationale behind its choice is the same as discussed above for the Beta distribution. The present state of the knowledge of volcanic activity suggests that the magnitude of the eruptions probably follows a power-law relationship (e.g. Simkin & Siebert 1994; Pyle 1998). The parameters contained in α_4 have to account for this information (Marzocchi *et al.* 2004). It should be noted that the *a priori* distribution accounts for a possible relationship between repose time and size of a volcanic eruption only by setting a negligible probability of occurrence for eruptions with VEI ≤ 2 (a closed conduit regime implies a long period of repose). More quantitative and detailed formulation between repose times and the size of eruptions can be adopted once there is a general agreement on their validity. At this stage we suggest that this possible relationship should be taken into account by selecting different datasets to obtain different *a posteriori* distributions (see below); in this way it is possible to quantify the influence of the inclusion or removal of this information on the results (see Marzocchi *et al.* 2004).

(3) Through Bayes theorem and adopting the results of the conjugate families for the multinomial model, we obtain the following posterior distribution for θ_4:

$$\Theta_4 \mid Y_4,\ Y_3 \sim \mathrm{Di}_3(\alpha_4 + y_4) \tag{17}$$

where $\alpha_4 + y_4 = (\alpha_4^3 + y_4^3,\ \alpha_4^4 + y_4^4,\ \alpha_4^{5+} + y_3 - y_4^3 - y_4^4)$ and y_4^j is the number of eruptions of magnitude j, with $j = 3, 4$ observed in the set of y_3 eruptions that occurred at the previous node.

An example of long-term volcanic hazard assessment

To illustrate the long-term volcanic hazard assessment for a generic explosive volcano, we provide a hypothetical example consisting of the evaluation of the probability of eruption in a given time window τ. Let:

- $Y_1^N \mid \theta_1 \sim \mathrm{Bin}(1,\ \theta_1)$ be the variable that counts if a new episode of unrest occurs in a given time window τ, where θ_1 is the probability of unrest;
- $Y_2^N \mid Y_1^N = 1,\ \theta_2 \sim \mathrm{Bin}(1,\ \theta_2)$ be the variable that counts if the unrest is due to a magmatic intrusion, provided the unrest happened, where θ_2 is the conditional probability of magmatic intrusion;
- $Y_3^N \mid Y_2^N = 1,\ \theta_3 \sim \mathrm{Bin}(1,\ \theta_3)$ be the variable that counts if a new eruption occurs in a given time window τ, provided the magmatic intrusion happened, where θ_3 is the conditional probability of eruption.

With this formulation, the eruption event in a given time window τ can be written as ($Y_1^N = 1$, $Y_2^N = 1$, $Y_3^N = 1$); that is, an eruption occurs when simultaneously an unrest is detected, a magmatic intrusion is detected, given the unrest has occurred, and an eruption is detected, given the magmatic intrusion occurred. Alternatively, if unrest has already been detected, the eruption event in a given time window τ can be written as ($Y_2^N = 1$, $Y_3^N = 1 | Y_1^N = 1$). That is, an eruption occurs, given the unrest happened, when simultaneously a magmatic intrusion is detected, provided the unrest occurred, and an eruption is detected, provided the magmatic intrusion occurred.

When our hypothetical volcano is in a quiet period, we are interested in the long-term volcanic hazard assessment, whereas if the unrest is detected we are interested in the mid- to short-term volcanic hazard assessment. Here, we focus our attention only on the long-term volcanic hazard assessment.

Let us suppose that we wish to evaluate the probability of an eruption in a given time window τ, taking into account the known history of this hypothetical volcano. Let us suppose that, at this stage of knowledge, we start from a uniform distribution for the first three nodes ($\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = \alpha_3 = \beta_3 = 1$), and in the past we observe $y_1 = 8$ onsets of unrest out of $n_1 = 40$ time windows investigated, with evidence of $y_2 = 4$ magmatic intrusions and $y_3 = 1$ eruption. For simplicity, we indicate with P both the probability function (for a discrete random variable) and the density function (for a continuous random variable).

Then the probability we are looking for is

$$\begin{aligned}
&P(Y_1^N = 1, Y_2^N = 1, Y_3^N = 1 \mid Y_1, Y_2, Y_3) \\
&\quad = \int P(Y_1^N = 1, Y_2^N = 1, Y_3^N = 1 \mid \theta_1, \theta_2, \theta_3) \\
&\qquad \times P(\theta_1, \theta_2, \theta_3 \mid Y_1, Y_2, Y_3) \mathrm{d}\theta_1 \mathrm{d}\theta_2 \mathrm{d}\theta_3 \\
&\quad = \int P(Y_1^N = 1 \mid \theta_1) P(\theta_1 \mid Y_1) \mathrm{d}\theta_1 \\
&\qquad \times \int P(Y_2^N = 1 \mid Y_1^N = 1, \theta_2) P(\theta_2 \mid Y_2, Y_1) \mathrm{d}\theta_2 \\
&\qquad \times \int P(Y_3^N = 1 \mid Y_2^N = 1, \theta_3) P(\theta_3 \mid Y_3, Y_2) \mathrm{d}\theta_3 \\
&\quad = \int \theta_1 P(\theta_1 \mid Y_1) \mathrm{d}\theta_1 \int \theta_2 P(\theta_2 \mid Y_2, Y_1) \mathrm{d}\theta_2 \\
&\qquad \times \int \theta_3 P(\theta_3 \mid Y_3, Y_2) \mathrm{d}\theta_3 \\
&= E_{\Theta_1 \mid Y_1}(\theta_1) E_{\Theta_2 \mid Y_2, Y_1}(\theta_2) E_{\Theta_3 \mid Y_3, Y_2}(\theta_3) \\
&= \left(\frac{\alpha_1 + y_1}{\alpha_1 + \beta_1 + n_1}\right)\left(\frac{\alpha_2 + y_2}{\alpha_2 + \beta_2 + y_1}\right) \\
&\quad \times \left(\frac{\alpha_3 + y_3}{\alpha_3 + \beta_3 + y_2}\right) \\
&= \left(\frac{1+8}{1+1+40}\right)\left(\frac{1+4}{1+1+8}\right)\left(\frac{1+1}{1+1+4}\right) \\
&= 0.036 \qquad (18)
\end{aligned}$$

the product of the posterior expected values of the conditional probability of the risky event at each node (see equations (8), (11) and (14) for the posterior densities involved, and equation (6) for the expected value of a Beta distribution).

Concluding remarks

One of the most relevant features of the Bayesian approach is that it allows the probability of a specific event to be a random variable characterized by a statistical distribution. The first and second moment of such a distribution can be associated with different types of 'uncertainties'. Specifically, the intrinsic unpredictability of the system (aleatoric) can be estimated by the expected value of the pdf, whereas the uncertainty related to our limited knowledge of the system (epistemic) can be naturally associated with the standard deviation of the pdf (i.e. to the dispersion around the expected value). Often, decision-making is at best based on a single value of the volcanic hazard. However, the Bayesian approach allows us to include our *a priori* belief about the probabilities at each node. In other words, the long-term volcanic hazard assessment of an eruption in the frequentistic approach is evaluated with the product of the maximum likelihood estimates of the conditional probabilities at each node, that is, the product of the proportions of events at each node. For example, $(y_1/n_1)(y_2/y_1)(y_3/y_2) = (y_3/n_1) = 1/40 = 0.025$. It should be noted that the Bayesian long-term volcanic hazard for an eruption (e.g. 0.036) is greater than the frequentistic estimate of probability (e.g. 0.025), because it includes the information about maximum ignorance expressed through the *a priori* uncertainties. The Bayesian long-term volcanic hazard for an eruption can be seen as the weighted average of the probability of an eruption with the posterior distributions of the probabilities of the risky events. The dispersion of the prior distributions at each node furnishes our 'degree of knowledge' for that stage of the volcanic process, and therefore it may guide future research with the aim of reducing epistemic uncertainties.

Further reading

Gelman *et al.* (1995) have provided a comprehensive introduction to Bayesian methods. Newhall & Hoblitt (2002) have described the use of the event tree in volcanology, particularly for short- and mid-term hazard assessments. Marzocchi *et al.* (2004) have presented an actual hazard assessment of Vesuvius, Italy, using methods such as those presented in this paper.

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