

A numerical study for convection in a cylindrical model with continuously varying viscosity

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Abstract

In the framework of a cylindrical symmetry model for convective motions in the asthenosphere, a new profile for the viscosity coefficient depending on depth is suggested here. The numerical elaboration of the above mentioned model leads to interesting results which fit well with experimental observations. In particular these continuously varying viscosity solutions probably describe the convective motions within the Earth better than simple constant viscosity solutions. Consequently the temperature values seem to be a realistic representation of the possible thermal behaviour in the upper mantle.

Key words *convection currents – viscosity*

1. Introduction

The study of convective motions is of remarkable interest to explain important aspects of geodynamic processes. The presence of currents in the mantle has been demonstrated by many arguments in several sectors of geophysics. A very important argument is the continental drift, as strongly implied by palaeomagnetic studies, and the closely-related supposition of a «mobile» and young ocean floor (plate tectonics). Where upward-moving limbs of convection cells diverge, tensional forces in the Earth lithosphere would be expected, and has frequently been used to explain rifts on continents and median valleys on oceanic ridges.

Several studies show that low degree harmonics of the Earth gravitational field are be-

lieved to be more likely due to the pattern of convection currents rather than to inhomogeneities maintained by finite strength in the Earth. Furthermore, convection has frequently been invoked to explain the magnitude and distribution of heat-flow anomalies on oceanic ridges and other regions.

Another fact is the raising of the crust following the melting of the glaciers at the end of the ice ages, as for example the Scandinavia zone. These motions offer an indirect evidence of the material presence in the upper mantle which, under particular conditions of stress and temperature, has features very close to those of newtonian fluid and can be transported from one location to another. However, an analysis of these motions provides an estimate of the viscosity of the mantle underneath the crust. The Earth mantle has a rheologic behaviour which varies with respect to depth: the upper part, which shows a discontinuous thickness in different areas, is fundamentally rigid and

cohesive to the crust whereas the middle part (asthenosphere) has a somehow «fluid» behaviour. From this point of view the lithosphere is the upper part of a single external layer of the Earth, which is more or less rigid. In the layer just below the velocity of transverse waves is strongly attenuated. Since laboratory experiments show that seismic waves can be attenuated and absorbed by a liquid crystal mixture, it is reasonable to think that the asthenosphere behaves like a viscous fluid where convective motions exist.

The simpler model of convection motions is the Benard layer; consider a newtonian fluid confined between two horizontal planes separated by a distance. The fluid is heated from below and cooled from above, so that in equilibrium state of no motion a temperature gradient is maintained between the top and bottom boundaries. The motions have been widely studied both theoretically and experimentally (Turcotte and Oxburgh, 1972; Oxburgh and Turcotte, 1978). Conservation equations for mass, momentum, and energy are required. The viscosity ν , coefficient of thermal expansion α and thermal diffusivity K are taken to be constant; in the body force term of the momentum equation a linear force relation is assumed between the variations of temperature and density:

$$\rho - \rho_0 = -\rho_0 \alpha (T - T_0)$$

where T_0 is the temperature at which the density ρ is equal to the reference density ρ_0 . Introducing $\theta = T - T_0$ and $P = p + \rho_0 g z$ we can write the equations for conservation of mass, momentum, and energy, namely continuity, Navier-Stokes and energy equations,

$$\begin{aligned} \vec{\nabla} \cdot \vec{u} &= 0 \\ (\vec{u} \cdot \vec{\nabla}) \vec{u} &= -\frac{\vec{\nabla} P}{\rho} + \nu \nabla^2 \vec{u} + \alpha \theta g \vec{i} \\ (\vec{u} \cdot \vec{\nabla}) \theta &= K \nabla^2 \theta. \end{aligned}$$

The condition that there be no flow through the boundaries requires that $v = 0$ at $y = 0$ and $y = d$. A second velocity boundary condition is required on the horizontal boundaries. These studies show that the equilibrium state in the fluid exists until a critical value of a parameters set in which temperature is fundamental: indeed, when this value is overcome a convective current starts up in the medium. Instability condition is characterized by an adimensional quantity that varies with respect to both fluid features and boundary conditions, namely the Rayleigh number.

Detailed discussions on convection in the mantle can be found in the literature (Richter, 1978; Jarvis and Peltier, 1989; Peltier, 1989). The mathematical model that we consider in this work has already been introduced in the studies on «Earth dynamo» which try to explain the origin of the Earth magnetic field and the basic lines of its structure (Busse, 1970, 1975).

2. Discussion of the model

The convective model of the «Earth dynamo» is related to a substantially cylindrical geometry for the Earth nucleus. The same approach can be used for the whole globe, if we think of it as a rotating sphere, with a kind of approximation that wouldn't take polar caps into consideration.

The cylindrical model that has been considered seems to be an interesting one if examined between the two Tropics (fig. 1). Convection is bounded within the external ring that rotates around its own symmetry axis; at a first approximation, convection appears to be characterised by concentric circles having an axis parallel to the rotating one, insofar as the small inclination ψ of the upper and lower limits of the ring ensures that the Proudman-Taylor Theorem (PTT) proves to be valid (Taylor, 1974).

We consider the constraint imposed by rotation upon convection of a Boussinesq fluid driven by buoyancy forces.

The motion is governed by Navier-Stokes equations, adding terms considering the Earth

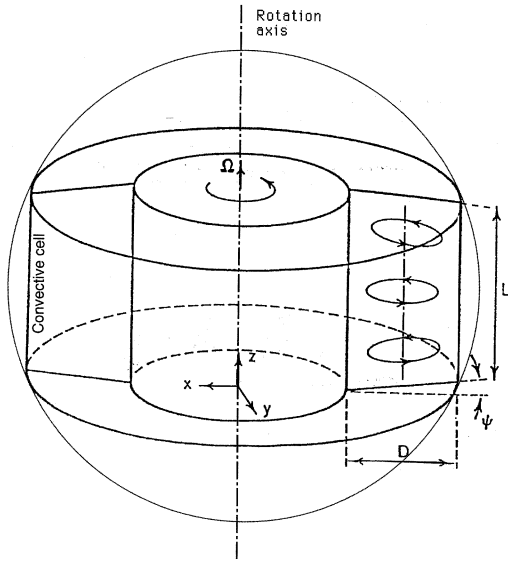


Fig. 1. Geometry of the rotating annulus model.

rotation effect

$$\begin{aligned} \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{u} + 2\Omega \vec{k} \times \vec{u} &= \\ &= -\frac{\vec{\nabla} P}{\rho} + \nu \nabla^2 \vec{u} + \alpha g \theta \vec{i} \\ \vec{\nabla} \cdot \vec{u} &= 0 \end{aligned}$$

$$T = T_0 - \beta \left[-x + \left(\frac{U}{\Omega} \right) \epsilon \right]$$

where T_0 is a constant, β is the temperature gradient, U is a typical fluid velocity and the dimensional less perturbation temperature, ϵ , is zero in the absence of motion. If we let $\vec{u} = U \hat{u}$ be the fluid velocity and adopt L and Ω^{-1} as our units of length and time respectively, the linear equation governing marginal convection is (Busse, 1978)

$$\frac{\partial \vec{u}}{\partial t} + 2 \vec{k} \times \vec{u} = -\vec{\nabla} \tilde{P} + E \nabla^2 \vec{u} + B \theta \vec{i}$$

where

$$\tilde{P} = \frac{P}{\rho L \Omega U}, \quad E = \frac{\nu}{\Omega L^2}, \quad B = \frac{\alpha \beta g}{\Omega^2}.$$

The boundary condition that the normal velocity vanishes implies that, since P and θ are almost independent of z the result is

$$\left(\frac{\partial}{\partial t} - E \nabla^2 \right) \omega - B (\vec{j} \cdot \vec{\nabla}) \theta = -4\eta (\vec{u} \cdot \vec{i})$$

where

$$\omega = \vec{k} \cdot \vec{\nabla} \times \vec{u} \quad \text{and} \quad \eta = \tan \psi \ll 1.$$

The governing equation admits separable solutions and yields the following algebraic equations, with \hat{P} and $\hat{\theta}$ integration constants,

$$(-iS + Ea^2) \hat{P} + 2i\alpha\beta \hat{\theta} = -4i\alpha\eta \hat{P}$$

$$\left(-iS + \frac{E}{P} a^2 \right) \hat{\theta} = \frac{1}{2} i\alpha \hat{P}$$

where

$$S = \frac{4\eta}{1+P} \frac{\alpha}{a^2} \quad \text{and} \quad a^2 = \gamma^2 + \alpha^2.$$

In this case the viscous dissipation is overcome by boundary forces when the Burger number is

$$B = P \left(\frac{a}{\alpha} \right)^2 \left[\frac{16\eta^2}{(1+p)^2} \left(\frac{\alpha^2}{a^4} \right) + \frac{E^2}{P^2} a^4 \right].$$

When L/D is so large that $E(L/D)^2 \gg P\eta/(1+P)$, where D is the unit of height, we have

$$B = \frac{E^2}{P} \frac{a^6}{\alpha^2}$$

since $R = P/E^2B$

$$R = \frac{a^6}{\alpha^2}.$$

In the case of both free surfaces, the critical value of Rayleigh number is

$$R_c = \frac{27\pi^4}{4} \left(\frac{L}{D}\right)^4$$

corresponding to

$$a_c^2 = \frac{3}{2} \left(\frac{\pi L}{D}\right)^2 \quad \text{and} \quad \alpha_c^2 = \frac{1}{2} \left(\frac{\pi L}{D}\right)^2$$

then

$$R_c = 658 \left(\frac{L}{D}\right)^4$$

In the case of one free and one rigid surfaces (Fanucci and Santini, 1995), the critical value of Rayleigh number becomes

$$R_c = 1100 \left(\frac{L}{D}\right)^4.$$

3. Convection in rotating sphere

To have a complete identification with the annulus model shown before, we have to write the Burger number and the Ekman number in

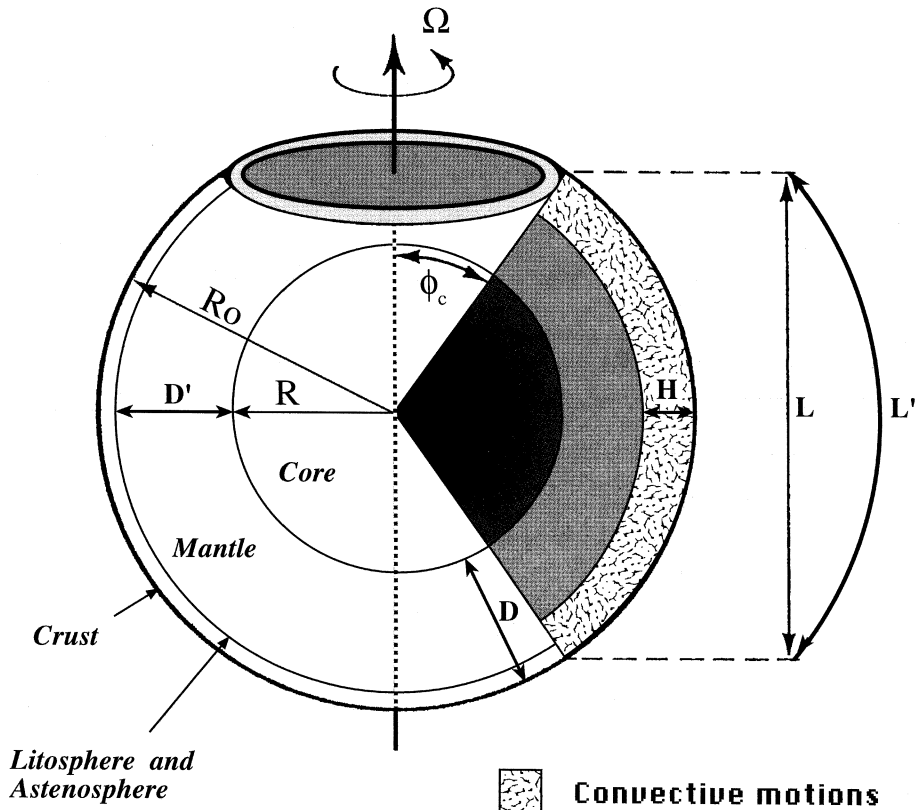


Fig. 2. Convection in a truncated rotating sphere. In this model the preferred mode is a motion confined in a thin annulus (H is its maximum thickness) surrounding the cylinder shown in fig. 1.

the following way

$$B = B_0 \sin^2 \phi \quad \text{and} \quad E = E_0 / (4 \cos^2 \phi)$$

where

$$E_0 = \nu / [\Omega (2R_0)^2]$$

$$B_0 = (\alpha \beta g_0) / \Omega^2$$

$$\vec{g} = -g_0 (\vec{r} / R_0)$$

then we obtain

$$B_{0c} = \frac{3}{\sin^2 \phi} \left[4PE_0^2 \left\{ \frac{\tan \phi}{(i+P) \cos \phi} \right\}^4 \right]^{\frac{1}{3}}$$

The B_{0c} absolute minimum is defined by the value $\phi = \phi_c$ and these become when $\sin \phi_c = 1/\sqrt{5}$, i.e.

$$\phi_c = 27^\circ.$$

Considering the previous treatment, the sphere can be approximate as a rotating annulus in which D is defined by the ray R_0 inclined by 27° with respect to the polar vertical axis. Now we must subtract the lithosphere and asthenosphere thickness from D to study the mantle convection. Furthermore we must multiply L by the complementary angle of ϕ , that is about $\frac{3}{8}\pi$, to have the real Earth circumference arc (Cathles, 1975):

$$L' = 2R_0 \frac{3}{8} \pi = 15000 \text{ km}$$

$$D' = 2710 \text{ km.}$$

The new Rayleigh critical number will be

$$R'_c = 1100 \left(\frac{L'}{D'} \right)^4 \approx 10^6.$$

We have, by Rayleigh critical number definition,

$$2 \cdot 10^{-6} z^4 / \nu > R'_c \rightarrow \text{convection}$$

$$2 \cdot 10^{-6} z^4 / \nu < R'_c \rightarrow \text{no convection}$$

and then

$$z \approx 560 \text{ km.}$$

If we call H the thickness in which the convective motions existence is proved, we can obtain by adding z to the lithosphere and asthenosphere thickness and subtracting the mean crust thickness from the result (fig. 2):

$$H = (560 + 175 - 35) \approx 700 \text{ km.}$$

Following these considerations we built a convective cell, with an approximately rectangular section, the sides of which are respectively 700 km and 2000 km (a value reasonable enough to involve oceanic ridges as ascending points).

In this case that basic equations of the model (Houston and De Bremaecker, 1975) have been conveniently adapted to the stated modifications; the energy equation (the heat transfer equation) and the momentum equation (the Navier-Stokes equation) lead to a system of partial differential equations which are strongly

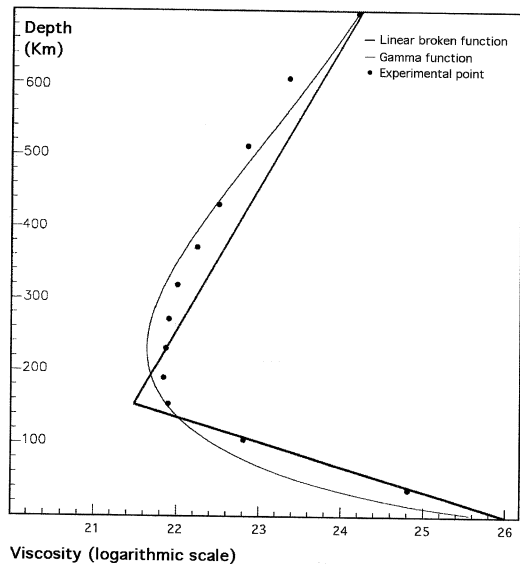


Fig. 3. Viscosity as a function of depth: points represent experimental data; solid lines show two selected functions approximating the viscosity behaviour.

coupled. For numerical integration and drawing of isothermal lines (which is the final result) we have used library routines (namely the Runge-Kutta method for resolution of differential equations and the Paw graphic software)

adapted through FORTRAN. The results obtained at a first numerical approach by linearly varying viscosity on depth with two different slopes (that is obtaining a broken line: see fig. 3) seem to be satisfactory; this model, with

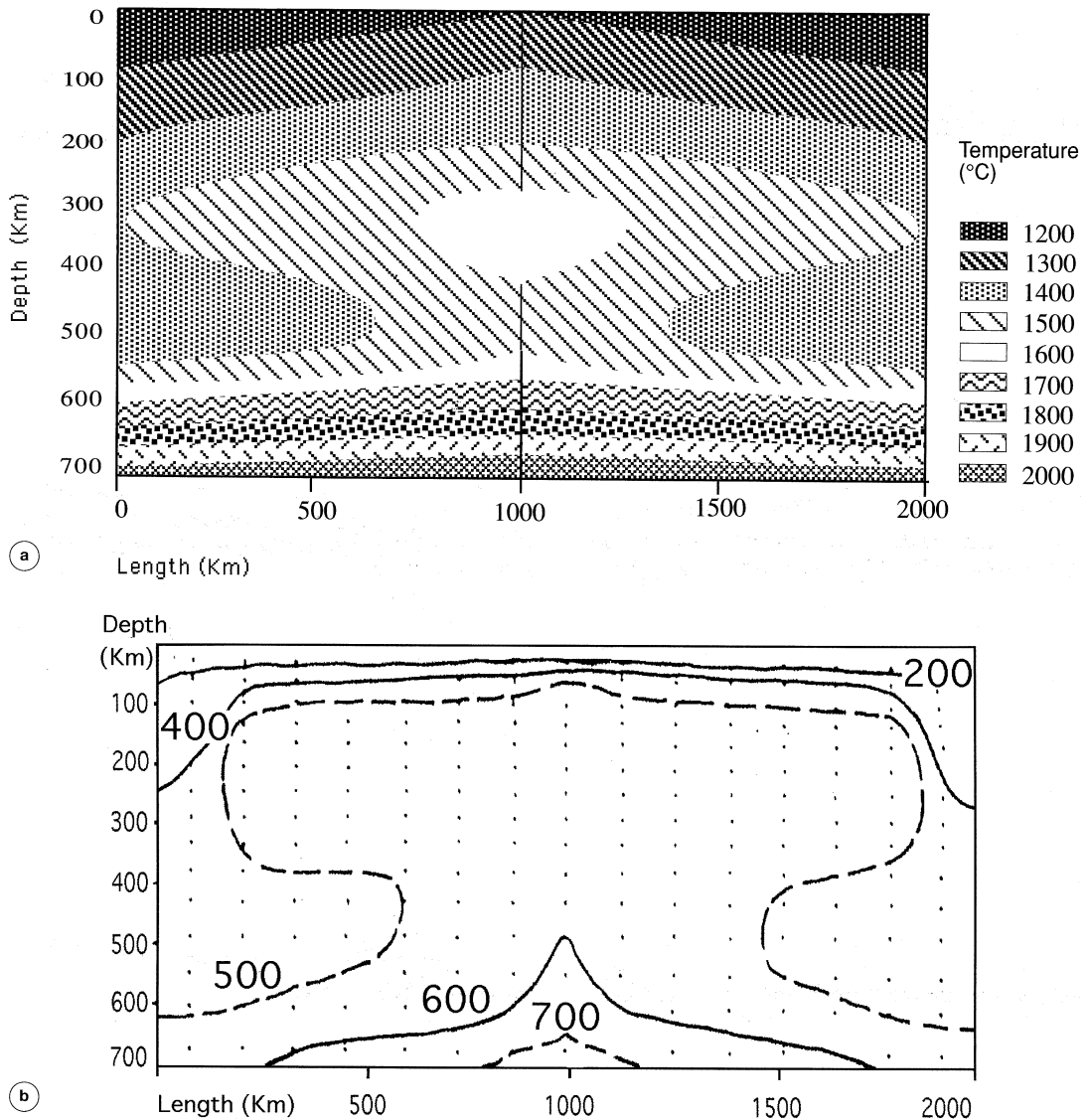


Fig. 4a,b. a) Convective cell temperatures in the $\Gamma(x)$ continuously variable viscosity case with bottom heat flux. b) Convective cell temperatures in the constant viscosity case with bottom heat flux (after Houston and De Bremaecker, 1975).

a heat flow coming from the bottom, displays very close isotherms in the region near the bottom of the convective cell, but this situation is not found on the layer just under the surface (Oxburgh and Turcotte, 1978).

With respect to the previous results, we have now obtained a better approximation by varying viscosity with depth according to a Gamma function distribution suitably translated according to experimental data:

$$\Gamma(x) = c(d-x)^\delta [M - (d-x)]^\sigma$$

where

$$\begin{aligned} c &= 28/10^{20} \\ d &= 2000 \\ x &= \text{depth} \\ M &= 2000 \\ \delta &= 5.4 \\ \sigma &= 0.7 \end{aligned}$$

The plot in fig. 3 shows this (better) fit to known values; in particular, it removes the unreasonable break at the transition between lithosphere and asthenosphere.

4. Conclusions

From a strictly geodynamic point of view the results obtained in this study show the following major points (fig. 4a):

a) the numerical model in which viscosity varies according $\Gamma(x)$ suggests that the lateral expansion of ascending hot material may be produced too deep to significantly affect the plates motion;

b) the high temperature body (the central area that is not hatched in fig. 4a), is situated at the minimum depth of 280 km and detached from the asthenosphere base. Leaving apart other considerations, this element could represent a feeding zone of an active hot «body»;

c) a continuously variable viscosity solution

correctly maintains symmetry in relation to the central line of circulation. This performance has already been showed in constant viscosity studies (Mckenzie *et al.*, 1973, 1974), fig. 4b; but the temperature profiles of our present model provide a more realistic representation of the upper Earth mantle.

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