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# The effect of magnitude uncertainty on the Gutenberg–Richter *b*-value estimation and the magnitude–frequency distribution: 'what hump?'

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## SUMMARY

The uncertainties on magnitudes of the earthquakes have a negligible effect on the estimation of the Gutenberg–Richter *b*-value if these uncertainties have a homogeneous distribution, that is, the magnitude error is the same for all the earthquakes. Here, we show that a non-uniform error distribution can have a significant impact on the *b*-value estimation, and it generates a hump in the magnitude–frequency distribution. Through a simulation approach, we show when the bias in the estimation is large, when it can be neglected, and how it is possible to avoid it.

**Key words:** Probability distributions; Statistical methods; Statistical seismology.

## INTRODUCTION

The distribution of the earthquakes' magnitude is well described by the Gutenberg–Richter law (Gutenberg & Richter 1944); represented by the following equation:

$$\mathrm{Log}\,(N) = a - b\,(M - M_{\min}) \quad (1)$$

where Log is the base-10 logarithm, $N$ is the total number of events above magnitude $M$, $M_{\min}$ is the completeness magnitude of the seismic catalogue, $a$ and $b$ are the parameters that represent the intercept and the slope of the theoretical line in a Y-log-scale plot; the parameter $b$ is usually called *b*-value.

The magnitude of the earthquakes, as all the measures, is affected by errors. Previous works demonstrated that the estimation of the *b*-value is not biased in the case of homogeneous magnitude error, that is, the uncertainty on the magnitude is the same for all earthquakes (Tinti & Mulargia 1985; Marzocchi & Sandri 2003). On the contrary, the estimation of the $a$ parameter is strongly biased, due to the nature of the Gutenberg–Richter law: since the number of small events is greater than the number of large events, among all the earthquakes recorded above the completeness more of the magnitude estimates are biased upwards than downwards.

Other works considered the more general case of non-uniform magnitude error: Kijko (1988) developed a method to estimate the *b*-value in the case where the magnitude error is expressed by an interval; Rhoades (1996) analysed the case of a normally distributed magnitude error. These works showed that the *b*-value estimation can be biased if the errors in the magnitude are non-uniform. Both methods applied an elegant iterative procedure to obtain an optimal estimate of the b-value.

Such iterative procedures become computationally expensive if we are interested in a temporal investigation or spatial mapping of the *b*-value, when we have to estimate hundreds or thousands of times different *b*-values.

The goal of this short paper is to show in which cases the bias in the *b*-value estimation due to the magnitude error is large, when it can be neglected, and how it is possible to avoid it. Through an intensive simulation approach, it is also shown the effect of the non-uniform magnitude error on magnitude–frequency distribution: the deep investigation of this secondary effect was not performed in the works of Kijko (1988) and Rhoades (1996).

## METHOD

### Magnitude simulation with error

To simulate the magnitudes affected by the error, we used the following equation:

$$M_i = M_0 + \mathrm{exp}_i + \mathrm{err}_i \quad (2)$$

where $M_i$ is the $i$th simulated magnitude, $M_0$ is the starting value for the magnitudes of the simulated catalogue, $\mathrm{exp}_i$ is an exponential random variable with parameter $\beta = b\ln(10)$, $\mathrm{err}_i$ is a Gaussian random variable with mean = 0 and standard deviation $=\sigma_i$ (Rhoades 1996; Marzocchi & Sandri 2003). Then, in this eq. (2), $M_0 + \mathrm{exp}_i$ represents the $i$th simulated magnitude conforming to the Gutenberg–Richter law and $M_i$ represents the simulated magnitude perturbed by an observation error $\mathrm{err}_i$.

### *B*-value estimation

To estimate the *b*-value of the simulated catalogues, we used the classical maximum likelihood estimation method of Aki (1965),

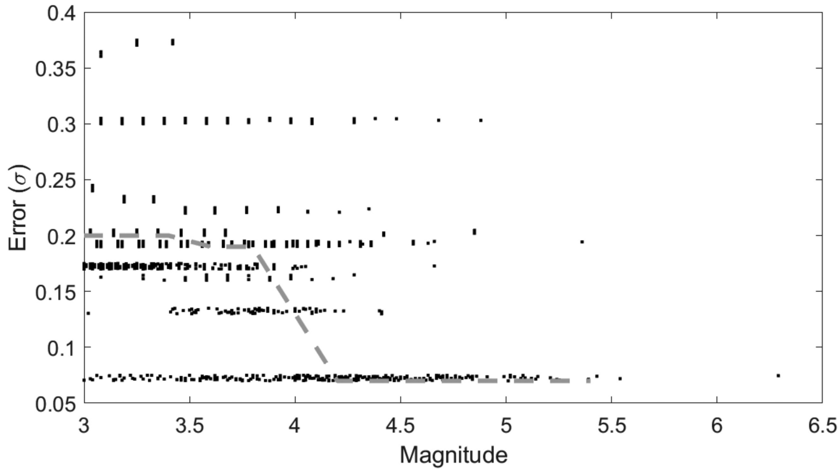


**Figure 1.** Magnitude versus error plot (black dots) for the HORUS catalogue from 2000 to 2019. Grey dashed line represents the median of the errors for each magnitude interval ([3.0,3.2), [3.2,3.4), etc.). To avoid dots overlap, a small noise was added to each error.

**Table 1.** Parameters for all the simulated catalogues.

| Catalogue name | $M_{\mathrm{Thresh}}$ | $\sigma_1$ | $\sigma_2$ |
|---|---|---|---|
| Sim no. 1 | 1.05 | 0.20 | 0.10 |
| Sim no. 2 | 1.10 | 0.20 | 0.10 |
| Sim no. 3 | 1.50 | 0.20 | 0.10 |
| Sim no. 4 | 2.00 | 0.20 | 0.10 |
| Sim no. 5 | 1.05 | 0.25 | 0.05 |
| Sim no. 6 | 1.10 | 0.25 | 0.05 |
| Sim no. 7 | 1.50 | 0.25 | 0.05 |
| Sim no. 8 | 2.00 | 0.25 | 0.05 |

with the correction for the binned magnitudes (Utsu 1966; Marzocchi & Sandri 2003) and the correction for an unbiased estimation (Ogata & Yamashima 1986; Marzocchi *et al.* 2020):

$$\hat{b} = \frac{\frac{n-1}{n}}{\ln(10)\left(\bar{M} - M_{\min} + \frac{\Delta M}{2}\right)} \quad (3)$$

where $\hat{b}$ is the estimated b-value, $n$ is the total number of events in the catalogue, $\bar{M}$ is the average of the magnitudes, $M_{\min}$ is the minimum magnitude of completeness and $\Delta M$ is the width of a magnitude bin (in our case of simulated magnitudes $\Delta M$ is zero).

## Data

To show a real example of a catalogue with different magnitude errors, we used the instrumental HORUS catalogue (Lolli *et al.* 2020) from 2000 to 2019, selecting the events with magnitude $\geq 3.0$ and depth $\leq 20$ km.

# RESULTS

To understand the effect of the magnitude error in the *b*-value estimation and on the magnitude–frequency distribution, we generated multiple simulated catalogues using eq. (2) to produce the magnitudes. In these simulations, we assumed that the error on the magnitude is non-uniform; in particular, we assumed a dichotomy in the error distribution. Until a certain magnitude threshold $M_{\mathrm{Thresh}}$, we used a Gaussian error with standard deviation $\sigma_1$, above $M_{\mathrm{Thresh}}$ we used a standard deviation $\sigma_2$. This simulation scheme, which might seem too simple, tries to emulate the behaviour of real catalogues. Indeed, some earthquake catalogues are built by merging different types of magnitudes (e.g. $M_{\mathrm{L}}$ and $M_{\mathrm{w}}$) characterized by errors of different amplitudes, and the proportion of these different magnitudes changes very rapidly after a certain magnitude threshold. Herrmann & Marzocchi (2021) clearly showed this rapid change for a set of catalogues for southern California. Therefore, we think that our scheme, although simple, can be considered a good proxy for real catalogues. As an example, Fig. 1 shows the magnitude vs error plot for the Italian instrumental catalogue HORUS (Lolli *et al.* 2020): around magnitude 4 it is possible to note the rapid decrease in the magnitude error. We underline that, in the case of catalogues with different types of magnitudes, also other problems in the *b*-value estimation can arise, for example, due to non-correct $M_{\mathrm{L}}$–$M_{\mathrm{w}}$ conversion (Herrmann & Marzocchi 2021).

We performed eight different types of simulations, each one composed of $10^4$ catalogues with $10^4$ events. All the events are generated using eq. (2) with a *b*-value equal to 1.00. Table 1 shows the different parameters used in each simulation. Each simulation starts from magnitude $M_0 = 0$, and the *b*-value estimation starts from $M_{\min} = 1.00$. The starting magnitude value $M_0$ of the simulations must be lower with respect to $M_{\min}$: when a magnitude is simulated we have to consider also the case of a magnitude initially lower than $M_{\min}$ that became larger than $M_{\min}$ after the error is added (Marzocchi *et al.* 2020). We underline that all the results obtained for $M_{\min} = 1.00$ are valid for all the other magnitude of completeness since the translation of $M_{\min}$ does not affect the shape of the magnitude–frequency distribution nor the *b*-value estimation if we assume the Gutenberg–Richter law (e.g. Taroni 2021).

In Figs 2 and 3, we showed the distribution of the estimated *b*-values (left-hand panels; dashed black line represents the true *b*-value), and the incremental magnitude distribution (right-hand panels, stacked for all the $10^4$ catalogues), for different values of $M_{\mathrm{Thresh}}$, $\sigma_1$ and $\sigma_2$. As magnitude errors $\sigma_1$ and $\sigma_2$, we used the following values: 0.20 and 0.25 for $\sigma_1$, 0.10 and 0.05 for $\sigma_2$; these values are in line with the real magnitude error for $M_{\mathrm{L}}$ and $M_{\mathrm{w}}$, respectively (e.g. Gasperini *et al.* 2013; Lolli *et al.* 2020 for the Italian instrumental seismic catalogues).

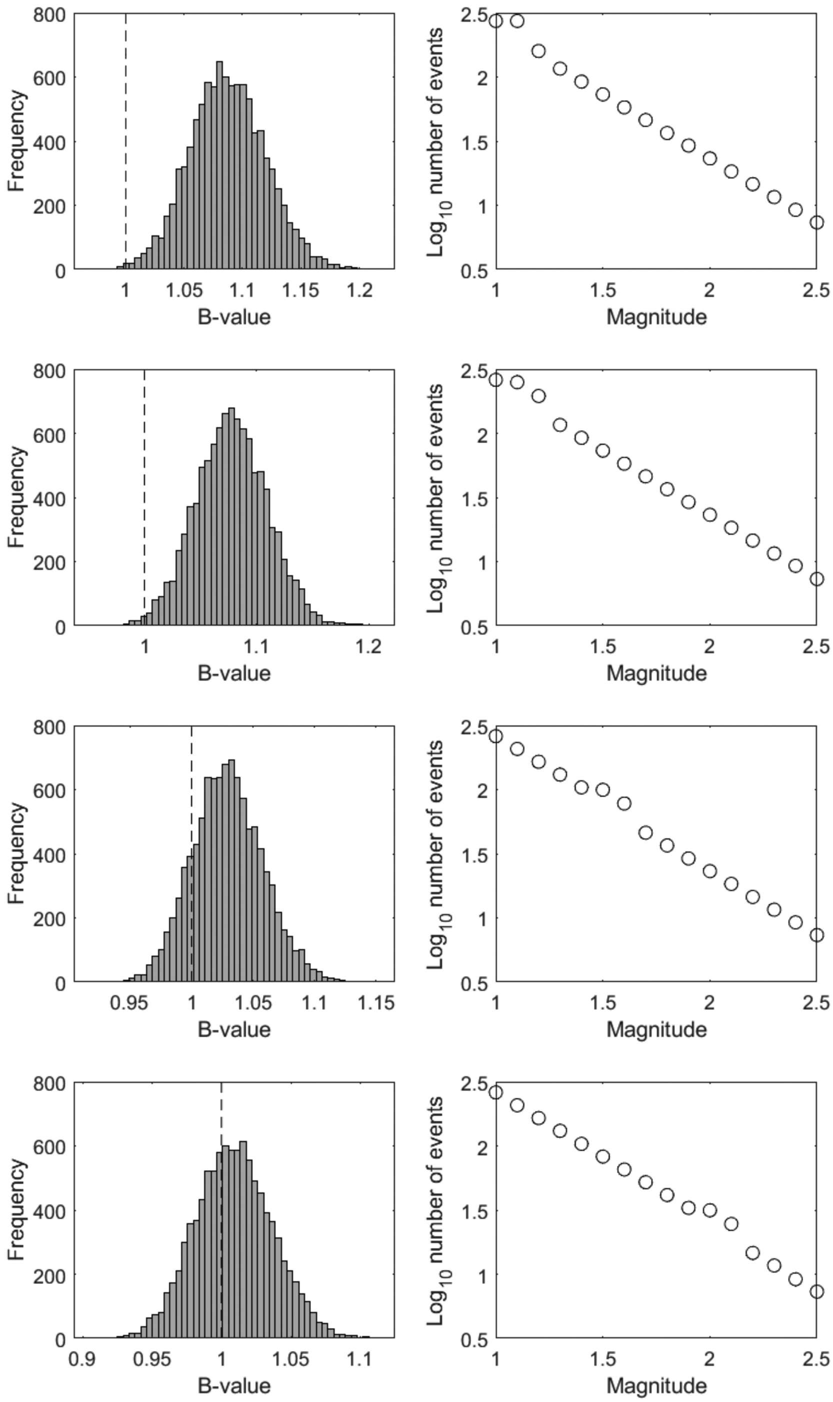


**Figure 2.** Histograms of the estimated *b*-values (left-hand panels) along with the *b*-value used for the simulations (vertical black dashed line), and magnitude–frequency distribution plots (right-hand panels, black circles indicate the incremental number of events in each magnitude bin); panels (a) and (b) for catalogue Sim no. 1, panels (c) and (d) for catalogue Sim no. 2, panels (e) and (f) for catalogue Sim no. 3 and panels (g) and (h) for catalogue Sim no. 4.

From the left-hand panels of Figs 2 and 3, we can appreciate the effect of the errors in the *b*-value estimation: the smaller the difference $M_{\mathrm{Thresh}} - M_{\mathrm{min}}$ the larger the bias. If the magnitude threshold $M_{\mathrm{Thresh}}$ is near to the magnitude of completeness, that is, $M_{\mathrm{Thresh}} - M_{\mathrm{min}}$ is 0.1 or 0.05, the bias is about + 10 per cent in the case of $\sigma_1 = 0.20$ and $\sigma_2 = 0.10$, and is about + 15 per cent in the case of $\sigma_1 = 0.25$ and $\sigma_2 = 0.05$. On the other hand, if the magnitude threshold $M_{\mathrm{Thresh}}$ is far from the magnitude of completeness, that is, $M_{\mathrm{Thresh}} - M_{\mathrm{min}}$ is 1.0, the bias is less than + 3 per cent.

These results suggest that the *b*-value estimation can be considered unbiased if $M_{\mathrm{Thresh}} - M_{\mathrm{min}} > 1.0$: such a rule can be very useful for all the seismologists interested in spatial or temporal variations of the b-value.

Regarding the magnitude–frequency distribution, the right-hand panels of Figs 2 and 3 clearly show an artificial hump in the incremental distribution of the magnitudes for all the cases at the value of the *X*-axis corresponding to $M_{\mathrm{Thresh}}$. Therefore, these simulations demonstrate that a hump in the magnitude–frequency distribution

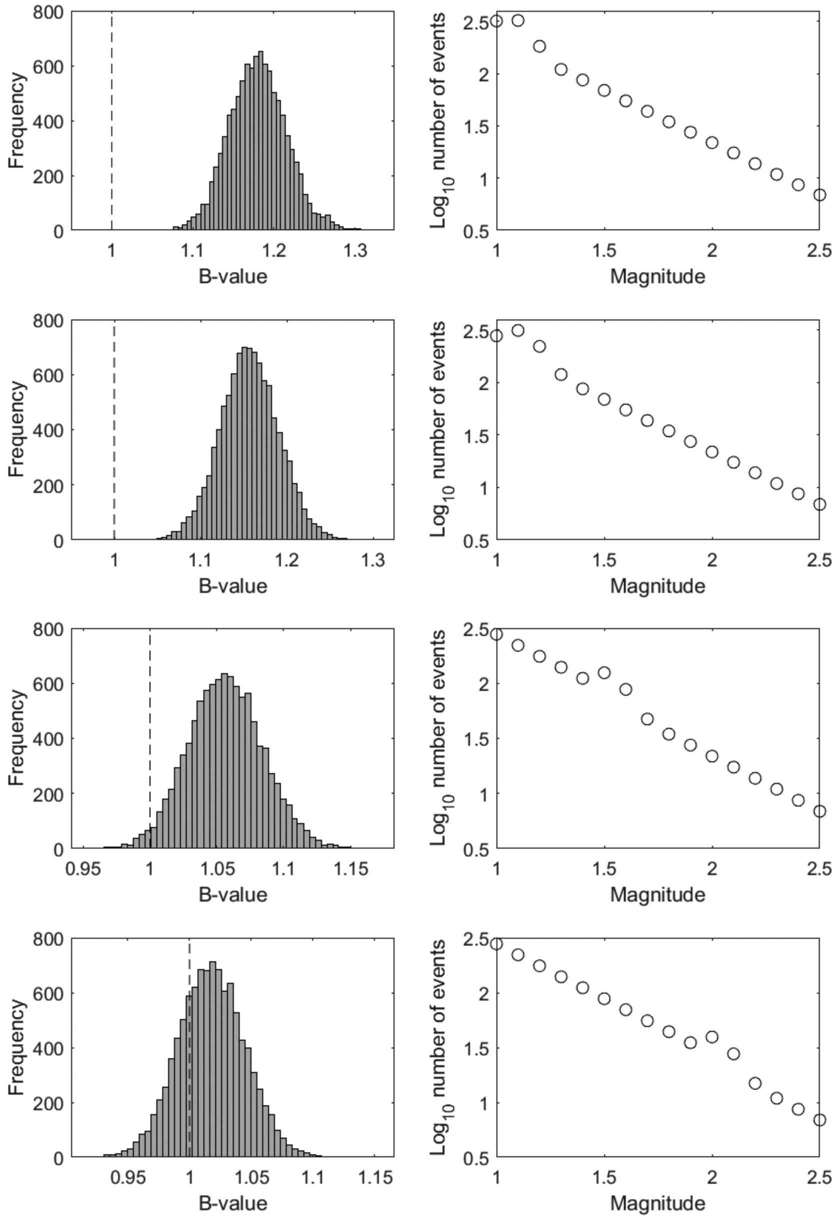


**Figure 3.** Histograms of the estimated $b$-values (left-hand panels) along with the $b$-value used for the simulations (vertical black dashed line), and magnitude–frequency distribution plots (right-hand panels, black circles indicate the incremental number of events in each magnitude bin); panels (a) and (b) for catalogue Sim no. 5, panels (c) and (d) for catalogue Sim no. 6, panels (e) and (f) for catalogue Sim no. 7 and panels (g) and (h) for catalogue Sim no. 8.

(also called 'bulge', Field *et al.* 2009) can be an artefact caused by non-uniform magnitude errors.

To investigate the existence of possible humps related to different magnitude errors in real catalogues, we also analysed the HORUS catalogue from 2000 to 2019 (Lolli *et al.* 2020). In Fig. 4, the black arrow indicates a hump in the incremental magnitude distribution for the magnitude bin 4.4 (i.e. $4.35 \le M_w < 4.45$). From Fig. 1, we know that near magnitude 4 there is a strong decrease in the median magnitude error (grey dashed line in Fig. 1), and between magnitude 4.2 and 4.5 the large majority of the errors became very small. Our previous simulated results suggest that, at least partially, the hump in bin 4.4 could also have been caused by the changes in the magnitude error. However, we underline that the natural variability in the number of events, especially in the right tail of the magnitude distribution like in the case of Fig. 4, can justify the observed hump. Therefore, in the 'Conclusions' section, we took into account only the results obtained through the simulations.

## CONCLUSIONS

In this short paper, we have shown how a non-uniform magnitude error affects the estimation of the Gutenberg–Richter $b$-value and the shape of the magnitude–frequency distribution.

Through an intensive simulation approach, we have built thousands of seismic catalogues with a non-uniform magnitude error: a Gaussian error with standard deviation $\sigma_1$ until a certain magnitude threshold $M_{\mathrm{Thresh}}$, and another Gaussian error with standard deviation $\sigma_2$ above $M_{\mathrm{Thresh}}$. Although simple, this error distribution tries

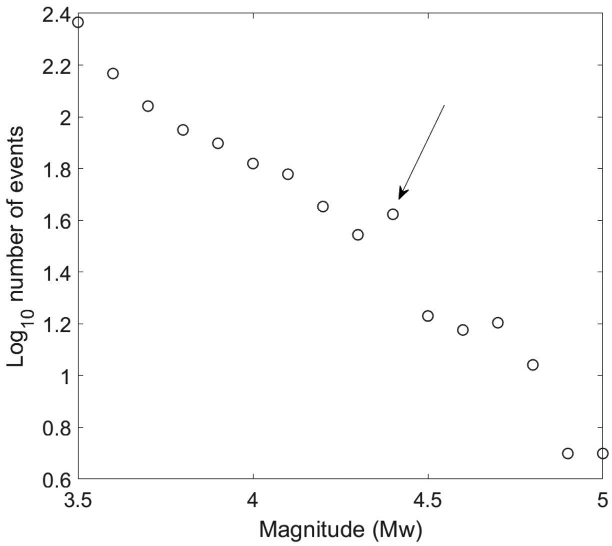


**Figure 4.** Magnitude–frequency distribution plot for the HORUS catalogue from 2000 to 2019 (black circles indicate the incremental number of events in each magnitude bin) in the magnitude range 3.5–5.0; the black arrow indicates a hump of this catalogue.

to emulate the behaviour of real catalogues (e.g. catalogue obtained by mixing $M_L$ and $M_w$ magnitudes).

From these simulated catalogues, we have obtained two important results:

(1) the *b*-value estimation is biased if $M_{Thresh}$ is near to the magnitude of completeness;

(2) the magnitude–frequency distribution shows a hump near $M_{Thresh}$.

The first result is in line with the works of Kijko (1988) and Rhoades (1996); the second result, that is, the artificial hump in the magnitude–frequency distribution (called 'hump' in honor of Marty Feldman/Igor of Young Frankenstein), is a novel outcome.

To avoid possible bias in the *b*-value estimation, we suggest paying attention to the magnitude error distribution in seismic catalogues. These findings also encourage the computation of the magnitude error (sometimes neglected) for all the seismic catalogues.

Possible future developments of this study can be the implementation of a more complex error structure and the inclusion of a potential magnitude binning, in order to consider seismic catalogues containing more than two magnitude types.

## DATA AVAILABILITY

The MATLAB code and data used in this paper are freely available at (last access June 2022): https://github.com/MatteoTaroniINGV/BvalueEstimationNonUniformMagnitudeError

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