Evolution of the number of communicative civilizations in the Galaxy: implications on Fermi paradox

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Abstract

It has been recently proposed DeVito (2019) that a *minimal* number of contacts with alien radio-communicative civilizations could be justified by their logarithmically slow rate of growth in the Galaxy. Here we further develop this approach to the Fermi paradox, with the purpose of expanding the ensemble of the possible styles of growth that are consistent with the hypothesis of a minimal number of contacts. Generalizing the approach in DeVito (2019), we show that a logarithmic style of growth is still found. We also find that a style of growth following a power law would be admissible, however characterized by an exponent less than one, hence describing a sublinear increase in the number of communicative civilizations, still qualitatively in agreement with DeVito (2019). No solutions are found indicating a superlinear increase in the number of communicative civilizations, following for example an exponentially diverging law, which would cause, in the long run, an unsustainable proliferation. Although largely speculative, our findings corroborate the idea that a sublinear rate of increase in the number of communicative civilizations in the Galaxy could constitute a further resolution of Fermi paradox, implying a constant and minimal - but not zero - number of contacts.

Keywords: Fermi paradox, Alien civilizations, Population dynamics

1. Introduction

DeVito DeVito (2019) has recently considered some new aspects of the "Fermi paradox", *i.e.,* the apparent contradiction between the lack of evidence for extraterrestrial civilizations existing in the Galaxy and their high probability Hart (1975); Webb (2002); DeVito (2013), suggested by the Drake equation Drake (2014); Forgan (2009). Assuming that the Galaxy is explored with the only purpose of detecting signals from alien radio-communicative civilizations, DeVito has argued that the rate R at which they are detected should depend on their number $n(t)$ but also on their rate of increase (or decrease), $\dot{n}(t)$. Note that here $n(t)$ represents the left-hand side of Drake's equation Burchell (2006); Sandberg et al. (2018), denoted by N and customarily assumed to be constant. A functional dependency like $R = R(n, \dot{n})$ appears to be justified, assuming an ideal scenario in which the Galaxy has been continuously explored during a significantly long period of time, taking note of the contacts with alien societies and continuing the search. Apart such idealized experiment, it seems clear that an explicit mathematical expression for the rate of detection can hardly be conjectured, although it seems reasonable to assume that R would be increasing with $n(t)$ and $\dot{n}(t)$. In general, the rate of successful detections shall depend upon the SETI strategy adopted, on the resources deployed, as well as on a number of other factors - also involving socio-political aspects - that can be hardly quantified lacking observational constraints.

Following DeVito, we make the hypothesis that n is large enough to be effectively treated as a continuous variable and that its time derivative $\dot{n}(t)$ can be evaluated for all values of t. Furthermore, assuming the functional dependency $R = R(n, \dot{n})$, the quantity

$$N^d = \int_0^T R(n, \dot{n})\, dt \tag{1}$$

represents the number of societies effectively detected over the exploration time interval $0 \leq t \leq T$. The argument in DeVito (2019) is that N^d cannot be a large number, otherwise some contact would have occurred by now. Since in the environment we have still not found evidence for such contacts (though search strategies for alien footprints have been suggested, see (Davies, 2012)), the DeVito's hypothesis is that N^d is small and *minimal.* This essential - although not verifiable - assumption, is the requisite for a quantitative approach to the problem, which otherwise would not be possible. Indeed, from functional analysis Kot (2014), for N^d being an extremum, $R(n, \dot{n})$ must obey the Euler-Lagrange (E-L) partial differential equation

$$\frac{\partial R}{\partial n} - \frac{d}{dt}\frac{\partial R}{\partial \dot{n}} = 0\,, \tag{2}$$

where henceforth we can assume $R \geq 0$ since R represents a rate of detection. Furthermore, a *necessary* condition for R being a minimum is

$$\frac{\partial^2 R}{\partial \dot{n}^2} \geq 0\,, \tag{3}$$

where $\dot{n}(t)$ is the time-derivative of the solution of Eq. (2). We note however that this constraint, known as "Legendre condition" in the calculus of variations (see *e.g.,* Gelfand and Fomin (1963)), has not been exploited in DeVito (2019). It is noteworthy that in the context of classical population dynamics, the introduction of variational principles dates back to the work of Volterra Volterra (1939), who considered the problem of minimizing an appropriate functional, leading to an E-L equation that is satisfied by the Verhlust (logistic) equation. The idea of Volterra proved to be fecund, being later reevaluated in Leitmann (1972) and Gatto et al. (1988).

Searching for a particular solution of the E-L equation (2) in the factorized form

$$R(n, \dot{n}) = G(n)H(\dot{n})\,, \tag{4}$$

where Lagrangian R is not explicitly time-dependent and the unknown functions $G(n)$ and $H(\dot{n})$ depend upon $n(t)$ and $\dot{n}(t)$ separately, DeVito DeVito (2019) has determined a *simple* solution of the problem, in which $H(\dot{n}) \approx \dot{n}^2$ (henceforth

$\approx$ is used to denote proportionality). With this choice, the minimum rate of detection turns out to be a constant, *i.e.,*

$$\dot{R}(n, \dot{n}) = 0 \,, \tag{5}$$

a condition that, by Occam's razor, appears to be reasonable and valid for any other acceptable solution of the E-L equation. According to DeVito, the solution $n(t)$ slowly increases with time following an unbounded logarithmic growth[1] (details shall be given in Section 2 below). Intriguingly, from this result DeVito has suggested a further possible resolution of Fermi paradox Webb (2002), *i.e.,* that the lack of contacts with alien communicative civilizations is hampered by their limited rate of growth in the Galaxy.

As emphasized in DeVito (2019), the solution of the E-L equation is, from a mathematical standpoint, highly non-unique. Furthermore, any solution could be hardly tested against experimental observations, at least until SETI shall succeed. Nevertheless, we think that searching and classifying other possible and yet unknown solutions of the DeVito's problem may constitute an interesting intellectual exercise. Indeed, their nature could provide new resolutions of Fermi paradox, either supporting or challenging that proposed in DeVito (2019). For instance, solutions characterized by a marked growth in time like $\sim \mathrm{e}^t$ or $\sim t^\alpha$ ($\alpha > 1$) would undermine DeVito's argument; *vice versa*, weakly increasing ($\sim t^\alpha$, $\alpha < 1$) or decaying solutions (as $\sim \mathrm{e}^{-t}$ or $t^{-\alpha}$ with $\alpha > 0$) would strengthen it. In this work we explore such possibilities, conventionally defining as *viable solutions* those for which Eqs. (2), (3) and (5) are simultaneously valid, as they are valid for DeVito's original logarithmic solution. Obviously, of particular interest are those viable solutions that can be expressed in terms of elementary functions, thus having a value similar to the *simple* solution sought (and found) in DeVito (2019). As far as we know, such alternatives have not been systematically explored so far. It is certain, however, that assuming for

[1] To avoid confusion, it is worth to remark that in population ecology the term *logarithmic growth* is used to indicate the phase of population growth during which the number of cells increases exponentially, in conditions of unlimited resources (see *e.g.,* Berryman (2003)).

$H(\dot{n})$ a degree three polynomial is not leading to viable solutions (see Appendix of DeVito (2019)).

This brief communication is organized as follows. In Section 2 we review and complement the DeVito's solution. In Section 3, we extend DeVito's solution scheme, obtaining a class of viable solutions characterized by logarithmic growth. Section 4 proposes a further viable and simple solution exhibiting a power law style of growth. Section 5 discusses the various styles of growth suggested by our results, which are compared with basic styles of growth known in the literature of population dynamics. Our conclusions are drawn in Section 6.

2. Extending DeVito's solution

DeVito DeVito (2019) relied upon the factorized form (4), in which $H(\dot{n})$ is the lowest-degree monomial expression for which a "simple" solution can be easily determined. Note that with respect to DeVito (2019), here we use a slightly different notation. Assuming

$$H(\dot{n}) = (c\dot{n})^2, \tag{6}$$

where c is a constant, and solving the E-L equation (2) by separating the variables we obtain

$$-2\frac{\ddot{n}}{\dot{n}^2} = \frac{G'(n)}{G(n)} = k^2, \tag{7}$$

where we have defined $G'(n) = \frac{dG}{dn}$ and k^2 is a dimensionless separation constant. Henceforth we assume, without loss of generality, that functions G and H are positive. The second of the two equalities in Eq. (7) gives $G(n) = G_0 \mathrm{e}^{k^2(n-n_0)}$, where $G_0 > 0$ is a constant and $n_0 = n(0)$ is the initial number of communicative civilizations, while from the first we obtain the following linear ordinary differential equation

$$\dot{n} = \frac{\dot{n}_0}{1 + \dfrac{\dot{n}_0 k^2}{2}\, t}\,, \tag{8}$$

where $\dot{n}_0$ is the initial rate of change of $n(t)$. Here we depart slightly from DeVito (2019), since we consider separately two cases that differ for the sign of the initial rate $\dot{n}_0$. Of course, according to (8), in the particular case $\dot{n}_0 = 0$, $n(t)$ would remain constant to n_0 during the whole observation period. By integrating (8) for $\dot{n}_0 \neq 0$, and defining a time constant τ such that $\tau^{-1} = |\dot{n}_0| k^2$, we obtain the time evolution of communicative civilizations that ensures an extremum for N^d, namely

$$n_{\pm}(t) = n_0 + 2\tau\,|\dot{n}_0| \log \left| \frac{t}{2\tau} \pm 1 \right| , \tag{9}$$

where $n_+(t)$ and $n_-(t)$ correspond to the two mutually excluding conditions $\dot{n}_0 > 0$ and $\dot{n}_0 < 0$, respectively.

In Figure 1, solutions (9) are qualitatively depicted for some particular values F1
of the free parameters; details are given in the caption. We note that solution $n_+(t)$ (red curve) corresponds to the one found in DeVito (2019). It is characterized by a slow unbounded growth and by a rate of change decreasing like t^{-1}, hence approaching zero for $t \mapsto \infty$. Although $n_-(t)$ (blue curve) is matching $n_+(t)$ for sufficiently long times ($t \gg \tau$), it appears that the sign of $\dot{n}_0$ has a significant role in shaping the solution for times $t \approx \tau$. Remarkably, Figure 1 shows that the condition of minimum for N^d (see Eq. 1) could be compatible with an initial decline and a subsequent recovery of the number of communicative civilizations, as indicated by solution $n_-(t)$. It should be observed, however, that according to our assumptions, $n(t)$ should be enough large to be considered as a real (and differentiable) variable, so that close to the singularity of Figure 1 the solution found has merely a formal character. It is straightforward to verify that the Legendre condition (3) is met for both $n_+(t)$ and $n_-(t)$, indicating that they could effectively correspond to a minimum of N^d. Note that the constraints represented by the Legendre condition has not been taken into consideration in DeVito (2019). In addition, the minimum rate of detection, *i.e.,* the value of $R(n, \dot{n})$ evaluated using for $n(t)$ the expressions of $n_{\pm}(t)$, is a constant (see 5). Hence, according to our definition of viable solution given above, the DeVito's solution and its extension (9) are both viable, being at the same time mathematically simple.

3. Generalizing DeVito's scheme

To better explore the range of possibilities existing, with the aid of the algebraic manipulator `Mathematica`® Research (2010), we have been searching for other viable and mathematically simple solutions of the E-L equation. In this section, we consider a few examples in which a factorized form (4) for $R(n, \dot{n})$ is preserved.

First, we have found that a straightforward generalization of DeVito's solution (9) is possible by making the particular choice

$$H(\dot{n}) = (c\dot{n})^p \,, \tag{10}$$

where c is an inessential constant and $p \geq 2$ is an integer (for $p = 2$, Eq. 10 reduces to 6). In this case, imposing the validity of the E-L equation (2), after some algebra we still find a logarithmic law

$$n_{\pm}(t) = n_0 + p\tau \, |\dot{n}_0| \log \left| \frac{t}{p\tau} \pm 1 \right| \,, \tag{11}$$

where constant τ and the meaning of $n_{\pm}(t)$ are the same of Eq. (9). It is easily verified that for even values of p the Legendre condition is satisfied, hence N^d could effectively have minimum for $n(t) = n_{\pm}(t)$. Conversely, for odd values of p, the Legendre condition only holds for $\dot{n} > 0$, hence, for $\dot{n} < 0$ the solution certainly does not correspond to a minimum. Note that similar to DeVito's solution, for $n = n_{\pm}(t)$ the rate of detection $R(n, \dot{n})$ is a constant. Hence, for even values of p, solution (11) is viable and characterized by the same level of mathematical complexity of (9). Figure 2 shows $n_+(t)$ for some even values F2 of p, using log-log axes. All the curves are similar to curve $n_+(t)$ in Figure (1), and regardless the p vale adopted their trends become distinguishable only for $t \geq \tau$. This example clearly supports the DeVito's argument about the logarithmic nature of the growth of $n(t)$. For $p \mapsto \infty$, it is easily verified that $n_+(t)$ approaches asymptotically the linear growth model $n(t) = n_0 + (\dot{n}_0\tau)(t/\tau)$, which is plotted by the purple curve in Figure 2.

By algebraic manipulation, we have found other interesting analytical solutions of the E-L equation. To provide a few examples, here we consider the three characterized by the simplest structure, namely $H(\dot{n})=(c_1\dot{n})\ \log(c_2\dot{n})$, $H(\dot{n})=c_1\dot{n}+(c_2\dot{n})^{-1}$ and $H(\dot{n})=\ \mathrm{e}^{c\dot{n}}$, where c_1, c_2 and c are positive constants. In the first case, for the time evolution of the number of communicative civilization we find

$$n_{\pm}(t) = n_0 + |\dot{n}_0|\tau \log \left| \frac{t}{\tau} \pm 1 \right| \,, \tag{12}$$

where constant τ and the meaning of $n_{\pm}(t)$ are the same as in Eq. (9). In the second case, after some algebra, we still find a solution that varies logarithmically with time, namely

$$n_{\pm}(t) = n_0 - |\dot{n}_0|\tau \log \left| \frac{t}{\tau} \mp 1 \right| \,, \tag{13}$$

whereas in the third case, we obtain

$$n(t) = n_0 + \frac{t}{\tau_1} + \tau_2 \left(\dot{n}_0 - \frac{1}{\tau_1} \right) \left(1 - \mathrm{e}^{-t/\tau_2} \right) \,, \tag{14}$$

where $\tau_1 > 0$ and $\tau_2 > 0$ are two independent time constants. We note that (12) and (13) confirm qualitatively the character of the original DeVito's solution (9). However, a qualitatively different style of growth is implied by (14), which shows, for sufficiently long times ($t \gg \tau_2$), a constant rate of increase, with $\dot{n}(t) \approx {\tau_1}^{-1}$. It is easy to establish, however, that all the three solutions considered above imply a time-varying minimum rate of detection ($\dot{R} \neq 0$), contrary to the original DeVito's solution (9) and to its extension (11). Hence, according to our conventions, they cannot be considered viable solutions.

4. More solutions

From the results so far, it appears that DeVito's hypothesis of a minimal number of detected civilizations suggests a logarithmic evolution for $n(t)$. As pointed out in DeVito (2019), it is of course impossible to scrutinize all the possible particular solutions of the E-L equation. However, either using an algebraic manipulator or by trial and error, we have made efforts to determine viable alternatives to the logarithmic growth that we have often encountered, hoping that in this way the *zoo* of possible solutions can be better explored.

Since the style growth (or decline) of a time-dependent function are commonly expressed terms of logarithms ($\log t$), exponentials ($e^{\alpha t}$) and powers (t^{α}), we have first searched for exponential solutions, but we have not been successful. Indeed, finding a solution characterized by a diverging exponential increase could be important, since this would challenge the results achieved in DeVito (2019) about the slowly growing number of radio-communicative civilizations in the Galaxy, assuming that the rate of detection is minimal. Similarly, for the same reason, the existence of a solution that grows according to a power law like t^{α} with $\alpha > 1$ would be engrossing, since it would influence the interpretation of Fermi paradox. We have not found viable solutions having a periodic character.

In our exploration, an interesting and surprisingly simple power-law solution for $n(t)$ has been found by trial and error assuming a rate of detection

$$R(n, \dot{n}) \approx n^p \, \dot{n}^q \,, \tag{15}$$

where $p \geq 2$ and $q \geq 2$ are free parameters. The form (15) appears meaningful, since it predicts a rate of detection that, for a given value of the number of societies $n(t)$, increases with their rate of change $\dot{n}(t)$, and *viceversa*; the values of p and q determine which of the two functional dependencies is stronger. We note, however, that Eq. (15) implies $R = 0$ if $n(t)$ is constant. Of course, p and q are *a priori* unconstrained, since we do not dispose of any experimental observation of R yet. Imposing the validity of the E-L equation (2), after some algebra we obtain a non-linear, autonomous ordinary differential equation in the unknown $n(t)$ that reads

$$p\,\dot{n}^2 + q\,n\,\ddot{n} = 0. \tag{16}$$

By direct substitution, it can be verified that (16) has a particular solution in the form of a power law

$$n(t) \approx \left(\frac{t}{\tau}\right)^{\beta} , \tag{17}$$

consistent with the initial condition $n_0 = 0$, where τ is a time constant, and where the exponent is

$$\beta = \frac{q}{p+q} \,. \tag{18}$$

We note that since $\beta < 1$ for any value of p and q, the growth of $n(t)$ is relatively slow and its rate is decreasing with time, never exceeding a linear trend. We remark that, based on our criteria, solution (17) is viable since *i)* it obeys the Legendre condition (3), and *ii)* the minimum rate of detection corresponding to the solution in Eq. (17) is a constant, according to (5).

5. Discussion

The existence of viable alternatives to the logarithmic model of growth, suggested by result (17), justifies a short discussion, in a broad perspective, about the significance of styles of growth encountered or simply mentioned in this work. It is convenient to classify them into two families, *i.e.,* superlinear and sublinear, according to the trend that they show in the long run, in comparison to a linear growth.

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Some examples of superlinear styles of growth are shown in the plot of Figure 3, where they are compared to the linear growth $n_{lin}(t) = t/\tau$ depicted by the dashed line. They are the exponential growth $e^{+\frac{t}{\tau}}$ (*i*, black curve), which exemplifies the Malthusian law of uninhibited growth known in population dynamics (Berryman, 2003), and two power laws with exponent $\alpha > 1$, *i.e.,* the quadratic (*ii*, $\alpha = 2$) and the cubic (*iii*, $\alpha = 3$) displayed in orange and red, respectively. In our exploration of the possible solutions of the E-L equation obeying the DeVito's hypothesis of a minimal number of detected civilizations, we have never encountered superlinear growth models like those considered in Figure 3. Of course, since our search cannot be exhaustive, the existence of admissible superlinear models is not ruled out. However, it seems unlikely that an exponentially diverging number of communicative civilizations may be compatible with the minimum (and constant) detection rate hypothesized in DeVito (2019). A common tenet in population dynamics is that an exponentially diverging growth would eventually become unsustainable and cause a collapse, analogous to the well known Malthusian catastrophe Malthus (1872). Along these lines, it is interesting to note that a "sustainability solution" to the Fermi paradox has been proposed in Haqq-Misra and Baum (2009), in which the absence of contacts is explained by the possible non sustainability of exponential (or faster) growth patterns of hypothetical intelligent civilizations.

As possible examples of sublinear styles of growth, in Figure 4 we have F4 considered the (shifted) logarithm $\log(1+t/\tau)$ (i, green curve), and two samples of power laws with exponent $0 < \beta < 1$, namely $(t/\tau)^{0.2}$ (ii, orange) and $(t/\tau)^{0.5}$ (iii, red). The dashed line still indicates the linear growth $n_{lin}(t) = t/\tau$. In Section 2, logarithmic solutions like (i), qualitatively similar to the one originally proposed by DeVito (DeVito, 2019) and encountered in this study, have been found to be in agreement with the E-L equation. Comparing the dashed curve with the green one, the sublinear character of the logarithmic growth is apparent although for times $t \ll \tau$ the two curves are matching. Similarly, in Section 3, we have shown that power-like styles of growth similar to those exemplified by (ii) and (iii) are admissible solution of the E-L equation (see, in particular, Eq. 17). We note that depending upon the value of exponent β, power-like sublinear growths can exceed the logarithmic one, as it is indeed the case in Figure 4 for $\beta = 0.5$ (iii). Both, however, remain strictly sublinear for $t \geq \tau$ and, *a fortiori*, sub-exponential.

It is worth to remark that, in our search of possible solutions to the DeVito's problem, we have not found examples of self-limiting patterns of growth that would eventually evolve to a constant value of $n(t)$, hence ultimately turning to sublinear and bounded styles of growth. This is characteristic of the very well known law in population ecology expressed by the logistic function first found by Verhulst (Berryman, 2003), and of other qualitatively similar models encountered in various fields like those of Gompertz (Zwietering et al., 1990), von Bertalanffy (Fabens et al., 1965), Beverton-Holt (Beverton and Holt, 2012) or Liquori and Tripiciano Liquori and Tripiciano (1980). All these sigmoidal growth models are characterized by a horizontal asymptote for long times, hence they are bounded (for a review, see Buis (2017)). As far as we now, a purely logarithmic unbounded growth like the one consistent with the DeVito's hypothesis of a minimal number of contacts, has never been proposed in the framework of population dynamics. Indeed, this could be partly due to the limited time period covered by the observations available (see *e.g.,* Bre), which hinders a precise assessment of a possible long-term asymptote. However, we note that Tanaka Tanaka (1982) has proposed a complex growth law of logarithmic nature to explain the life-lasting development of the size of certain mollusks (see also Ebert et al. (1999)). Similarly, we are not aware of the existence of theoretical growth models based on unbounded power laws with exponent less than one, which according to our results may constitute a solution of the DeVito's problem as well. It should be noted, however, that an unlimited growth resembling a power law has been observed in nature for certain secular trees Buis (2017).

6. Conclusions

Following DeVito's DeVito (2019) hypothesis of a constant and minimal rate of detection of communicative societies in the Galaxy, we have studied the general style of growth of such societies. Our results confirm that the logarithmic style of growth already proposed by DeVito (2019) would constitute a viable solution of the E-L equations. However, in this work, we have shown that a logarithmic solution would be also viable starting from more general Lagrangians DeVito (2019). Furthermore, by exploring the range of possible "simple" solutions of the E-L equations, we have found that styles of growth following a power law could be also compatible with DeVito's hypothesis, but only if characterized by an exponent less than one, hence by a decreasing rate of variation. Such possibility was not previously considered in DeVito (2019). No periodic, sigmoidal (*i.e.,* logistic) or exponentially diverging solutions seem to be compatible with DeVito's hypothesis. As proposed in Haqq-Misra and Baum (2009) in the context of Fermi paradox, these latter would be not sustainable in the long run.

Expanding the main result in DeVito (2019), our work suggests that a possible resolution of Fermi paradox is the slow, *sublinear growth* of the number of communicative civilizations in the Galaxy.

Acknowledgements

We thank Carl DeVito for discussion and advice. We also thank Francesco Mainardi and Gian Italo Bischi for constructive comments and continuous encouragement. GS is funded by a FFABR (Finanziamento delle Attività Base di Ricerca) grant of MIUR (Ministero dell' Istruzione, dell'Università e della Ricerca) and by a research grant of Dipartimento di Scienze Pure e Applicate (DiSPeA) of the Urbino University "Carlo Bo".

References

, . Population dynamics in benthic invertebrates. A virtual handbook. Version 01.2. http://www.thomas-brey.de/science/virtualhandbook. Accessed: October 2, 2019.

Berryman, A.A., 2003. On principles, laws and theory in population ecology. Oikos 103, 695–701.

Beverton, R.J., Holt, S.J., 2012. On the dynamics of exploited fish populations. volume 11. Springer Science & Business Media.

Buis, R., 2017. Biomathématiques de la croissance: Le cas des végétaux. EDP sciences.

Burchell, M.J., 2006. W (h) ither the Drake equation? International Journal of Astrobiology 5, 243–250.

Davies, P.C., 2012. Footprints of alien technology. Acta Astronautica 73, 250–257.

DeVito, C.L., 2013. Science, SETI, and mathematics. Berghahn Books.

DeVito, C.L., 2019. On the Meaning of Fermi's paradox. Futures , 389–414.

Drake, N., 2014. How my dad's equation sparked the search for extraterrestrial intelligence. National Geographic, 30 June 2014.

Ebert, T.A., Dixon, J.D., Schroeter, S.C., Kalvass, P.E., Richmond, N.T., Bradbury, W.A., Woodby, D.A., 1999. Growth and mortality of red sea urchins Strongylocentrotus franciscanus across a latitudinal gradient. Marine Ecology Progress Series 190, 189–209.

Fabens, A.J., et al., 1965. Properties and fitting of the von Bertalanffy growth curve. Growth 29, 265–289.

Forgan, D.H., 2009. A numerical testbed for hypotheses of extraterrestrial life and intelligence. International Journal of Astrobiology 8, 121–131.

Gatto, M., Muratori, S., Rinaldi, S., 1988. A functional interpretation of the logistic equation. Ecological Modelling 42, 155–159.

Gelfand, I., Fomin, S., 1963. Calculus of variations. 1963. Revised English edition translated and edited by Richard A. Silverman .

Haqq-Misra, J.D., Baum, S.D., 2009. The sustainability solution to the Fermi paradox. arXiv preprint arXiv:0906.0568 .

Hart, M.H., 1975. Explanation for the absence of extraterrestrials on Earth. Quarterly Journal of the Royal Astronomical Society 16, 128.

Kot, M., 2014. A first course in the calculus of variations. volume 72. American Mathematical Society.

Leitmann, G., 1972. A minimum principle for a population equation. Journal of Optimization Theory and Applications 9, 155–156.

Liquori, A., Tripiciano, A., 1980. Cell growth as an autocatalytic relaxation process, in: Vito Volterra Symposium on Mathematical Models in Biology, Springer. pp. 400–409.

Malthus, T.R., 1872. An Essay on the Principle of Population.

Research, W., 2010. Mathematica 8.0.

Sandberg, A., Drexler, E., Ord, T., 2018. Dissolving the Fermi Paradox. arXiv preprint arXiv:1806.02404 .

Tanaka, M., 1982. A new growth curve which expresses infinitive increase. Publications of Amakusa Maine Biological Laboratory 6, 167–177.

Volterra, V., 1939. Calculus of variations and the logistic curve. Human Biology 11, 173–178.

Webb, S., 2002. If the universe is teeming with aliens... where is everybody?: fifty solutions to the Fermi paradox and the problem of extraterrestrial life. Springer Science & Business Media.

Zwietering, M., Jongenburger, I., Rombouts, F., Van't Riet, K., 1990. Modeling of the bacterial growth curve. Appl. Environ. Microbiol. 56, 1875–1881.

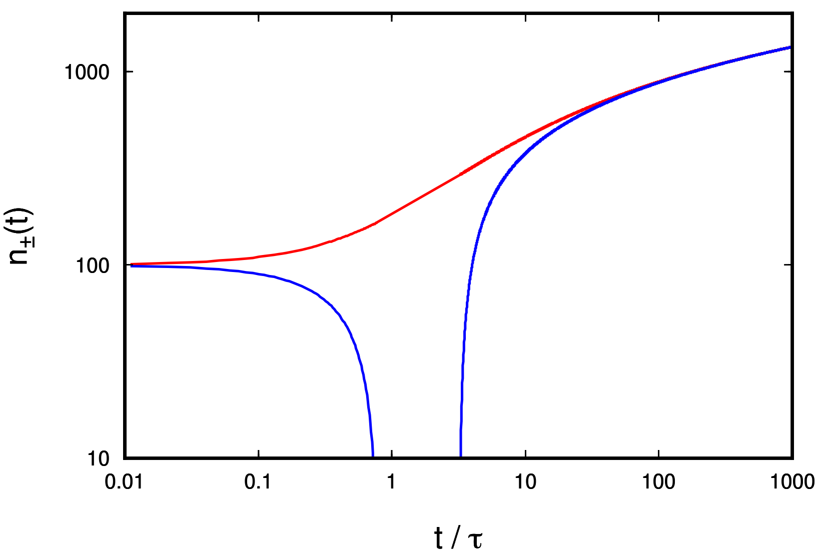

Figure 1: Solutions of the DeVito's problem, given by Eq. (9), for $n_0 = 100$ and $\dot{n}_0 \tau = 1$, as a function of the non-dimensional time t/τ, in a log-log plot. Red and blue curves correspond to solutions $n_+(t)$ and $n_-(t)$, respectively.

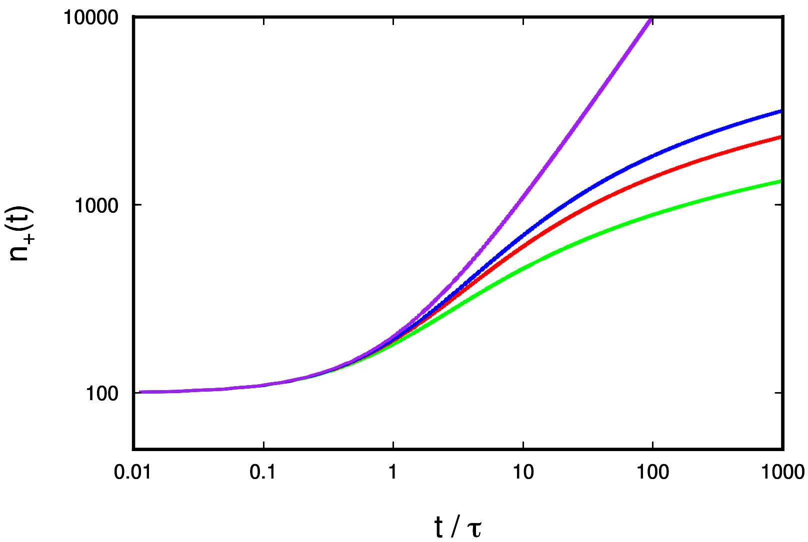

Figure 2: Plots of $n_+(t)$ according to Eq. (11), for $n_0 = 100$ and $\dot{n}_0\tau = 100$, as a function of t/τ, in a log-log plot. Green, red, blue, and purple curves correspond to values $p = 2$, 4, 6, and $p \mapsto \infty$, respectively.

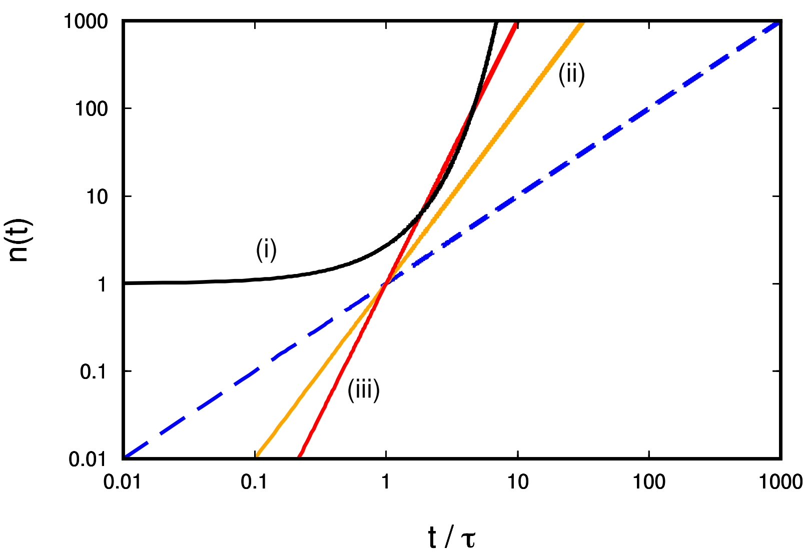

Figure 3: Number of communicative civilizations $n(t)$ according to a few hypothetical super-linear growth models. These include the positive exponential (*i*, black) and two samples of power-laws $(t/\tau)^{\alpha}$ with exponent $\alpha > 1$, $\alpha = 2$ (*ii*, orange) and $\alpha = 3$ (*iii*, red). The blue dashed curve shows, for reference, the linear growth. Since we are adopting a log-log scale, the power laws appear as lines with slopes increasing with α.

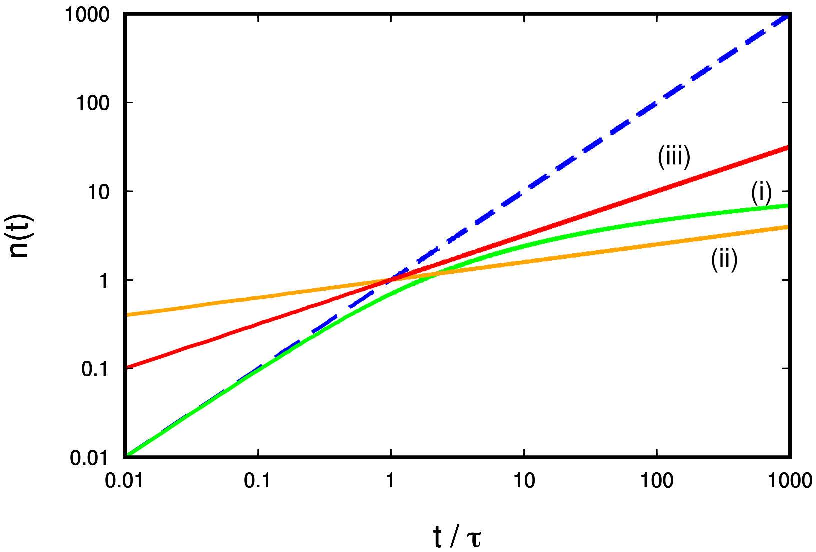

Figure 4: Number of communicative civilizations $n(t)$ according to various sublinear growth models of interest in this work. These include the logarithmic law (*i*, green), the power-laws $(t/\tau)^\beta$ with exponents $\beta = 0.2$ (*ii*, orange) and $\beta = 0.5$ (*iii*, red). The linear model is shown for reference by a dashed curve. Power laws appear as linear trends in this log-log plot.