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Back to the future: old methods for new estimation and test of the Gutenberg–Richter *b*-value for catalogues with variable completeness

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SUMMARY

In this short paper we show how to use the classical maximum likelihood estimation procedure for the *b*-value of the Gutenberg–Richter law for catalogues with different levels of completeness. With a simple correction, that is subtracting the relative completeness level to each magnitude, it becomes possible to use the classical approach. Moreover, this correction allows to adopt the testing procedures, initially made for catalogues with a single level of completeness, for catalogues with different levels of completeness too.

Key words: Statistical methods; Statistical seismology.

INTRODUCTION

The Gutenberg–Richter law (Gutenberg & Richter 1944) describes the distribution of the earthquakes' magnitude; and it is represented by the equation:

$$\mathrm{Log}\ (N) = a - b\,(M - M_{\min}), \qquad (1)$$

where Log is the logarithm with base 10, N is the cumulative number of events above magnitude M, $M_{\min}$ is the minimum magnitude of completeness of the seismic catalogue, a is the parameter that controls the number of events in the catalogue and b is the parameter that controls the slope of the line in a log-scale plot.

The parameter b is called *b*-value; this is one of the most studied seismic parameter: in the last decades about 7000 papers and posters, according to Google Scholar, have been published on it.

The classical method used to estimate the *b*-value is described by Aki (1965), and it is based on the maximum likelihood estimation (MLE) approach; the equation describing such estimator is:

$$\hat{b} = \frac{1}{\ln(10)\left(\bar{M} - M_{\min}\right)}, \qquad (2)$$

where $\hat{b}$ is MLE of the *b*-value, *ln* is the natural logarithm and $\bar{M}$ is mean of the magnitudes above $M_{\min}$ in the earthquake catalogue.

The eq. (2) cannot be used for catalogues with a spatio-temporal variation of the completeness magnitude: this constitutes a strong limitation of this approach considering that all the seismic catalogues usually present a variation of the completeness. Indeed, seismic catalogues usually have a completeness magnitude that decreases with time; moreover, just after the strongest events, an increase of the completeness magnitude affects both local (Lolli & Gasperini 2006) and global catalogues (Kagan 2003).

Kijko & Smit (2012) propose a way to generalize Aki (1965) approach for catalogues with different levels of completeness: they use the same MLE approach, and they demonstrate that the final estimation of the *b*-value for the whole catalogue is a combination of the *b*-values estimated in each subcatalogue with the same level of completeness. If $\hat{b}_1,\ \hat{b}_2,\ \ldots,\hat{b}_k$ are the Aki's MLE of the *b*-values for the k subcatalogues, $n_1,\ n_2,\ldots,\ n_k$ are the number of events in each subcatalogue and n is the total number of events in the whole catalogue, we have:

$$\hat{b}_{\mathrm{tot}} = \frac{1}{\frac{n_1/n}{\hat{b}_1} + \frac{n_2/n}{\hat{b}_2} + \ldots + \frac{n_k/n}{\hat{b}_k}}, \qquad (3)$$

where $\hat{b}_{\mathrm{tot}}$ is the final MLE of the *b*-value.

Despite its simplicity and its straightforward applicability, Kijko & Smit (2012) generalization of Aki (1965) approach is still not widely applied: a possible reason might be that, using this method, it is not easy to perform the testing procedure to compare the *b*-values that rely upon an underlying exponential distribution (e.g. Utsu 1966, 1999). Therefore, since most of the studies are focused on the comparison of the estimated *b*-values, a new method for such estimate and testing is desirable.

Another method useful to estimate the *b*-value in case of catalogues with time-varying magnitude of completeness is the Weichert (1980) approach. This method assumes a Poisson distribution of the seismic events and allows to compute the annual rate of events and the *b*-value at the same time using the MLE principle. Also in case of the Weichert (1980) approach, widely applied in seismic hazard computation and useful to estimate the *b*-value and the annual rate of events, it is difficult to use the classical testing procedure to compare the *b*-values.

In this short paper, we show a simple correction that allows to use the classical estimation and testing methods for catalogues

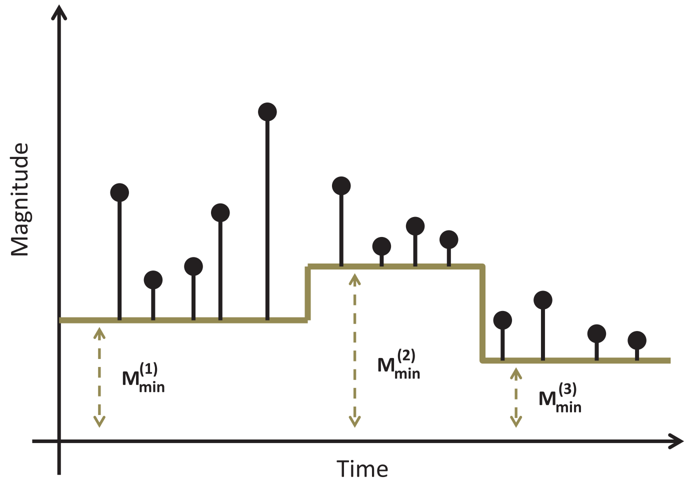

Figure 1. A schematic illustration of a catalogue with a time-varying magnitude of completeness $M_{\mathrm{min}}^{(k)}$.

Table 1. Magnitude of completeness in the synthetic catalogue.

Year	Completeness level
1960	4.0
1981	3.0
1990	2.5
2003	2.1
2005	1.8

with different levels of completeness: in particular the possibility of using the testing approach based on exponential distribution is an improvement respect both the Kijko & Smit (2012) approach and the Weichert (1980) approach.

METHOD

The Gutenberg–Richter law for the distribution of the magnitudes corresponds to an exponential distribution of the random variable $X_i = M_i - M_{\mathrm{min}}$ (Aki 1965), where M_i is the ith magnitude of the catalogue.

We found that it is possible to easily generalize this equation for catalogues with different levels of completeness in the following manner:

$$X_i = M_i - M_{\mathrm{min}}^{(k)}, \tag{4}$$

where $M_{\mathrm{min}}^{(k)}$ is the kth level for the magnitude of completeness associated to M_i (see Fig. 1).

Accordingly, this new random variable X_i follows an exponential distribution, and it is possible to apply the same MLE of Aki for the b-value. Considering eq. (4), now the eq. (2) became:

$$\hat{b}_{\mathrm{new}} = \frac{1}{\ln(10)\,\bar{X}}, \tag{5}$$

where $\bar{X}$ is the mean of the exponential random variable X_i. This eq. (5) exactly coincides with eq. (2) if $M_{\mathrm{min}}^{(k)}$ is constant.

It is important to note that, applying the correction in eq. (4), all the results obtained for the classical MLE of the b-value are still valid in case of catalogues with different levels of completeness: the estimation of the standard error $\hat{\sigma}_{\hat{b}} = \hat{b}/\sqrt{n}$ (Aki 1965), the correction for the binned magnitudes (i.e. $\frac{\Delta M}{2}$ added inside the parenthesis in the denominator, Utsu 1966; Marzocchi & Sandri 2003) and the correction for an unbiased estimation (i.e. $\frac{n-1}{n}$ in the numerator, Ogata & Yamashima 1986; Marzocchi *et al.* 2020).

Applying these last two corrections, and showing $M_{\mathrm{min}}^{(k)}$, eq. (5) became:

$$\hat{b}_{\mathrm{new}} = \frac{\frac{n-1}{n}}{\ln(10)\left(\frac{\sum_{i=1}^{n}\left(M_i - M_{\mathrm{min}}^{(k)}\right)}{n} + \frac{\Delta M}{2}\right)}, \tag{6}$$

where n is the total number of events in the catalogue and ΔM is the binning of the magnitudes (usually 0.1 for M_{L} and 0.01 for M_{w}).

We outline that this method assumes a constant b-value through the catalogue for all the different levels of completeness.

Another remarkable advantage of applying the correction proposed in eq. (4) is that the classical testing methods (e.g. Utsu 1966, 1999) based on the exponential distribution are still valid; these tests let to compare the b-values of two different groups, allowing to understand if a difference in two estimated b-values is statistically significant or not. Then the new method represents a significant improvement with respect to Kijko & Smit (2012) approach.

EXAMPLE

To provide an example of the application of the proposed method, we generated two synthetic seismic catalogues, catalogue A and catalogue B, sampling the magnitude from a Gutenberg–Richter law with b-value = 1 and b-value = 1.05, respectively. We mimicked the Italian instrumental seismic catalogue (Lolli *et al.* 2020), using the levels of completeness described in Table 1.

To do that, we selected from the whole catalogues (6×10^4 events each) only the events with a magnitude above the completeness level, obtaining a final catalogue A with 19 403 earthquakes (Fig. 2) and a final catalogue B with 19 055 earthquakes.

Then, we first checked if the random variable $X_i = M_i - M_{\mathrm{min}}^{(k)}$ was exponential, as expected by our approach. We used the Lilliefors test (Lilliefors 1967) for exponential distribution, test already applied for the earthquakes' magnitude (Marzocchi *et al.* 2020): we obtained a p-value = 0.66 for the catalogue A and a p-value = 0.73 for the catalogue B. The very high outcome values demonstrate that the hypothesis of exponential distribution of X_i cannot be rejected, that is our approach is robust.

Using eq. (6) and the estimation of the standard error $\hat{\sigma}_{\hat{b}} = \hat{b}/\sqrt{n}$ (that corresponds to the one obtained by Kijko & Smit 2012) we

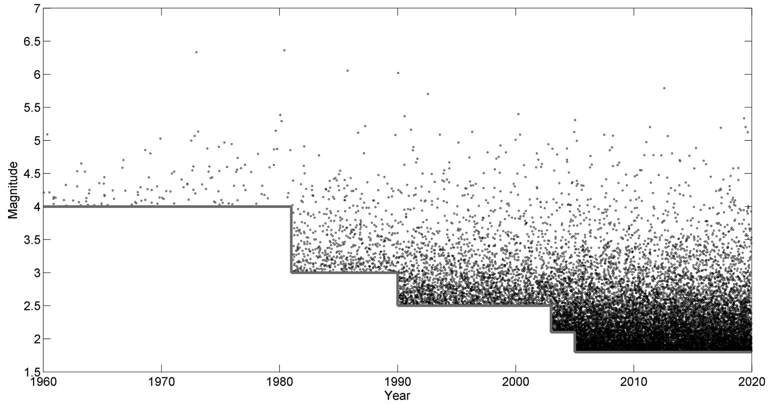

Figure 2. Year versus magnitude plot of the synthetic catalogue A; the black dots are the earthquakes, the grey line is the level of completeness that varies with time, according to Table 1.

get for catalogue A an estimated *b*-value $\hat{b}_A = 0.996$ (with $\hat{\sigma}_{\hat{b}} = 0.007$) and for catalogue B an estimated *b*-value $\hat{b}_B = 1.045$ (with $\hat{\sigma}_{\hat{b}} = 0.008$): these estimation completely agrees with the *b*-value used to generate the synthetic catalogues, demonstrating the validity of our approach.

Finally, we tested the hypothesis that the events in catalogue A and catalogue B are random samples from the same populations, that is from a Gutenberg–Richter law with a unique *b*-value, using the Utsu (1966) testing approach. Under this null hypothesis, $\hat{b}_B/\hat{b}_A$ follows the *F* distribution with $2N_A$ and $2N_B$ degrees of freedom, where N_A and N_B are the number of events in catalogue A and catalogue B, respectively (Utsu 1966). We obtained a *p*-value $= 1.25 \times 10^{-6}$: this very low value demonstrates, as expected, that the null hypothesis must be rejected and that $\hat{b}_B$ is significantly larger than $\hat{b}_A$.

CONCLUSIONS

In this short paper, we showed how to use the classical MLE for the *b*-value of the Gutenberg–Richter law for catalogues with a magnitude of completeness that varies with time. To apply our method, it is just necessary to correct the magnitude of the earthquakes by subtracting the relative magnitude of completeness. This very simple approach allows to use all the results related to the Aki (1965) MLE of the *b*-value, in particular the testing procedures that assume an exponential distribution of the magnitudes (Utsu 1966, 1999).

This method can improve the estimation and testing of the *b*-value, since it allows to use all the events above the different completeness levels in an earthquake catalogue. As example, this method can help the estimation of the *b*-value of an aftershock sequence (if we assume that the *b*-value is constant for the whole sequence), because usually after a strong main shock the magnitude of completeness is larger than the previous level (Kagan 2004).

Our approach, as the Aki (1965) MLE, assumes a Gutenberg–Richter law, that is an exponential distribution of the magnitude. Tapered or truncated versions of the Gutenberg–Richter law (Kagan 2002) may need a different type of estimation, especially in case of completeness level near the corner magnitude or the maximum magnitude (Marzocchi *et al.* 2020).

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