A Bayesian Framework for Rheology Model Combination and UQ in Simulation of Geophysical Mass Flows

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Motivation & Objective

- In hazard assessment for geophysical mass flows, we seek to construct accurate and reliable maps that show regions with high hazard.
- Acceptably accurate numerical simulation of complex geophysical mass flows is of crucial importance.
- Modeling mechanical behavior of such flows or the flow **rheology** presents a major difficulty.
- TITAN2D v. 4.0, the geophysical mass flow simulator, offers multiple well-known choices for flow rheology – Mohr-Coulomb, Pouliquen-Forterre, Voellmy-Salm.
- In this contribution, we present a Bayesian framework to combine the simulation results of alternative models and quantify the uncertainty in rheology models for both experimental and natural terrain flows.

1) Physics of Geophysical Flows

We assume the bulk mass of granular material as an incompressible continuum. Therefore, conservation of mass and momentum form the system of equations as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, \underset{\sim}{\mathbf{u}}) = 0 \quad (1)$$

$$\frac{\partial}{\partial t}(\rho\, \underset{\sim}{\mathbf{u}}) + \nabla \cdot (\rho\, \underset{\sim}{\mathbf{u}} \otimes \underset{\sim}{\mathbf{u}}) = \nabla \cdot \underset{\sim}{\mathbf{T}} + \rho\, \underset{\sim}{\mathbf{g}} \quad (2)$$

- Off-diagonal components of Cauchy stress tensor, $\underset{\sim}{\mathbf{T}}$, contain the flow rheology.
- We impose kinematic boundary conditions and apply "shallow-water" assumptions. This enables us to reduce the 3D problem to a 2D one by "depth-averaging" [1].

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h\bar{u}) + \frac{\partial}{\partial y}(h\bar{v}) = 0, \quad (3)$$

$$\frac{\partial}{\partial t}(h\bar{u}) + \frac{\partial}{\partial x}\left(h\bar{u}^2 + 0.5kg_z h^2\right) + \frac{\partial}{\partial y}(h\bar{u}\bar{v}) = S_x, \quad (4)$$

$$\frac{\partial}{\partial t}(h\bar{v}) + \frac{\partial}{\partial x}(h\bar{u}\bar{v}) + \frac{\partial}{\partial y}\left(h\bar{v}^2 + 0.5kg_z h^2\right) = S_y \quad (5)$$

Where:

- h, $h\bar{u}$ and $h\bar{v}$, are the flow thickness and depth-averaged flow momentum components along x and y directions.
- k, is the proportionality factor relating the in-plane normal stress components to the out-of-plane normal component.
- Source terms, S_x, S_y, contain the accelerating gravitational forces as well as the resisting forces due to the mechanical behavior and dynamics of flow.
- k, S_x and S_y, are specified by the rehology model.

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2) Bayesian Approach for Rheology Combination

- Let $(M_i)_{i=1,\dots,N_m}$ be a set of alternative rheology models developed to simulate the geophysical mass flows.
- In order to proceed this Bayesian method, the following materials form the set of necessary given data:
  1. At least one set of observation data, D, collected from a previous event (flow) at that site.
  2. An estimation for the initial conditions, $\mathcal{X}$, needed to simulate the event for which we have observation data.
- We can define the likelihood, $f(D, \mathcal{X}|M_i)$, using a proper ensemble of simulations for $\{D, \mathcal{X}\}$ given that model M_i was used.
- Avoiding any bias for the alternative models, we employ a prior discrete uniform distribution, such that $p(M_i) = \frac{1}{N_m}$, $_{i=1,\dots,N_m}$.
- We can consider the prior distribution as a set of weights for each rheology model.
- According to the Bayes' theorem, the posterior weights (discrete distribution) are calculated as:

$$w(M_i|D, \mathcal{X}) = \frac{f(D, \mathcal{X}|M_i)\; p(M_i)}{\sum_{j=1}^{N_m} f(D, \mathcal{X}|M_j)\; p(M_j)}$$

- Using these posterior weights, any estimator for some quantity of interest, η_i, could be combined through their Bayesian weighted average, $\bar{\eta}_b = \sum_{i=1}^{N_m} w(M_i|D, \mathcal{X})\; \eta_i$, which includes *model uncertainty*.

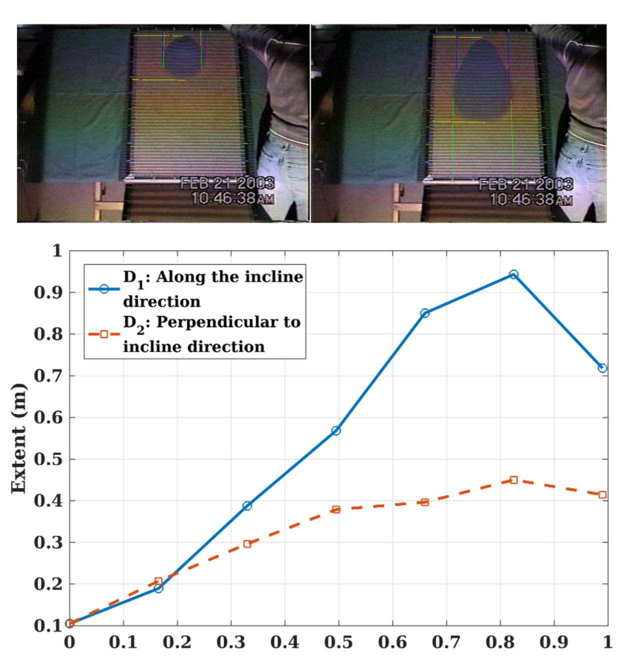
3) Metric-based Likelihoods

- Let $A^j(\mathcal{X})_{j=1,\dots,N_{samples}}$, be the simulated quantity and D be the corresponding observation data.
- Considering D, we define a proper "metric" or "distance function", $d(A^j(\mathcal{X}), D)$, for likelihood construction.
- We construct the likelihood, $f(D, \mathcal{X}|M_i)$, as:

$$f(D, \mathcal{X}|M_i) = P\left[d(A^j(\mathcal{X}), D) \le \epsilon | M_i\right], \quad _{i=1,\dots,N_m}$$

  Where ϵ, is a desirable threshold for metric values. Here, we choose the average of all samples.
- "Gaussian Stochastic Processes" allow the visualization of the metrics though high-resolution surrogates.

4) Flow Down an Inclined Ramp



Experimental setup specifications and I.C.s

Initial pile configuration	Cylindrical
Material weight	425 gr
Maximum pile height	6.1 cm
Major and minor extents of the pile	10.5 cm
Inclination angle	38.5°

$\Downarrow$

Considering all the simulation ensembles associated with alternative rheology models, μ_m, is the mean value of $\sum_{n=1}^{6} \frac{|D_{mn} - A^j_{mn}(\mathcal{X})|}{6}$.

$$d(A^j(\mathcal{X}), D_1 \cup D_2) := \frac{1}{2}\sum_{m=1}^{2}(1/\mu_m)\sum_{n=1}^{6}\frac{|D_{mn} - A^j_{mn}(\mathcal{X})|}{6}$$

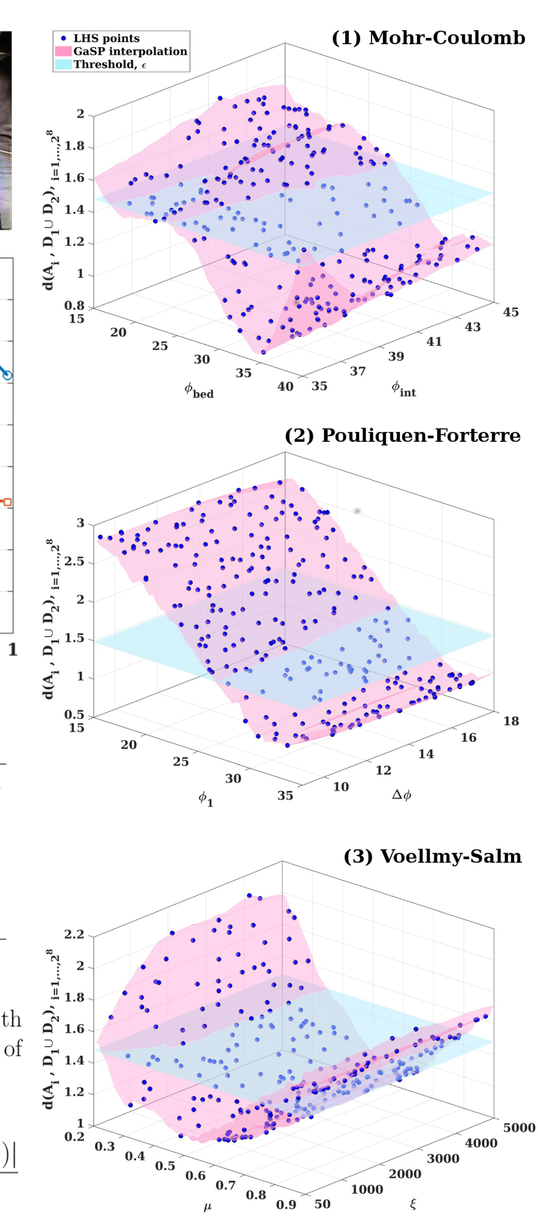


Table 1: Posterior weights regarding D_1, D_2 and $D_1 \cup D_2$.

Rheology Model:	$\{M_1, M_2, M_3\}$
$w(M_i\|D_1, \mathcal{X})$	$\{0.33, 0.28, 0.39\}$
$w(M_i\|D_2, \mathcal{X})$	$\{0.46, 0.24, 0.30\}$
$w(M_i\|D_1 \cup D_2, \mathcal{X})$	$\{0.35, 0.25, 0.40\}$

Conclusion

In the *hindcasting* cases (i.e., replicating previously occured flows) we described, this Bayesian scheme provided us valuable information on the combined-model performance. The results are useful tools for the future application of this method in construction of natural hazard *forecasting* associated with geophysical flows.

5) Bayesian Weighted Averaging for Flow Height Record Maps

The April 16-17, 1991, eruption of *Volcán de Colima*, Mexico, is a classical example of partial dome collapse with the generation of progressively long runout, Merapi-type pyroclastic flow (block-and-ash flow).

Table 2: Estimation for I.C.s, $\mathcal{X}$, Volcán de Colima 1991 eruption.

Pile center location (UTM East)	644956.0 m
Pile center location (UTM North)	2157970.0 m
Material volume	1.4×10^5 m^3

We adopted *Jaccard distance* as a proper metric:

$$d(A^j(\mathcal{X}), D) := \frac{(A^j(\mathcal{X}) \cup D) \backslash (A^j(\mathcal{X}) \cap D)}{A^j(\mathcal{X}) \cup D}$$

Here, D is the observed inundation area extent for this event and $A^j(\mathcal{X})$ is the inundated area obtained from each sample simulation.

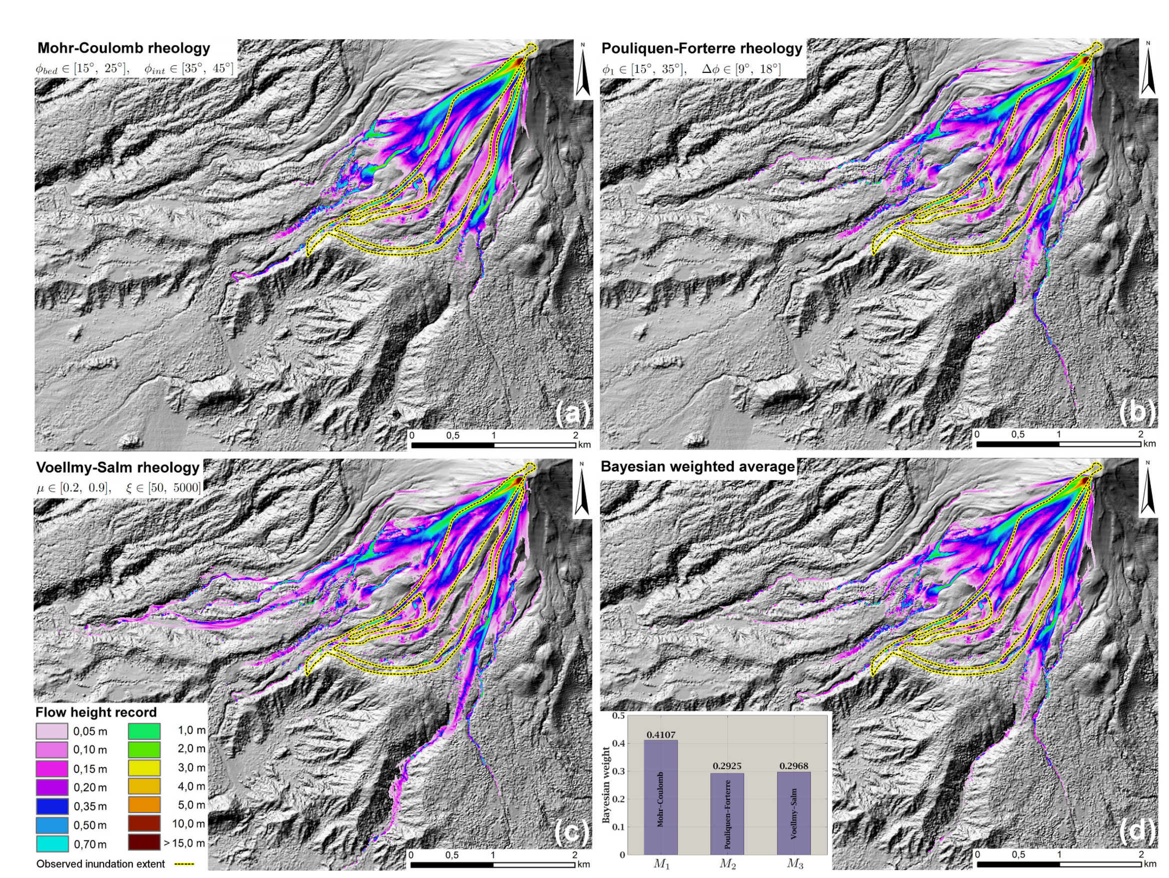


Figure 1: Flow height record maps considering single rheology models (a-c) and their Bayesian weighted average map (d).

6) Uncertainty Affecting Flow Height Record Maps

- Each rheology model produces a probability distribution for each specific target numerical value which can be expressed with mean and the *percentile values* of the combined-model results.
- Similar to what is done in "expert judgement" techniques, we treat the pool of rheology models, $(M_i)_{i=1,\dots,N_m}$, as a pool of experts – we combined the probability density distributions $(f_i)_{i=1,\dots,N_m}$ for the flow height record values [2].
- *Linear pooling* is a natural rule to define the combined-model percentiles [3]:

$$g(\cdot) = \sum_{i=1}^{N_m} w(M_i|D, \mathcal{X}) f_i(\cdot).$$

  Where the function $g(\cdot)$ is the probability mixture of the functions $(f_i)_{i=1,\dots,N_m}$.
- Linear pooling quantifies combined effects of the uncertainties in competing rheology models and their associated parameter spaces.

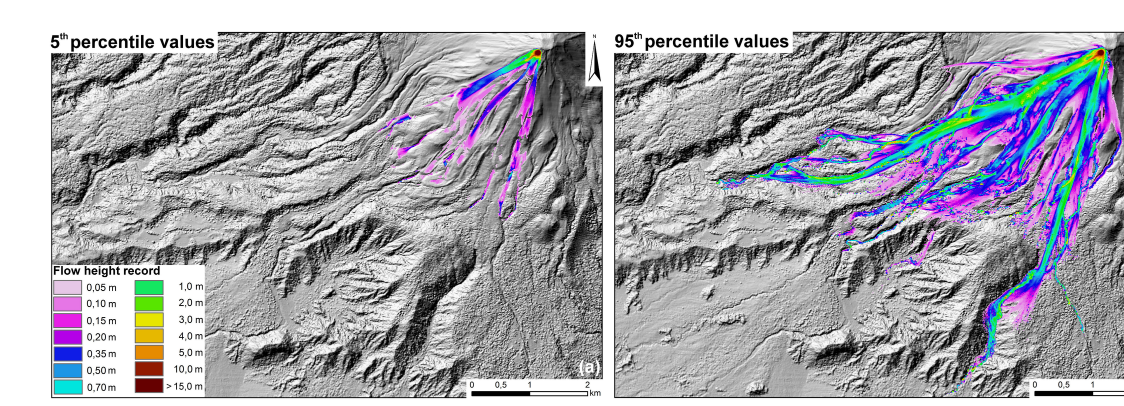


Figure 2: Following the linear pooling for $n = 2000$ times, the rheology model and then we sampled its specific parameters are randomly selected, providing a population of flow height values – the 5^{th} and 95^{th} percentiles of this population approximated the uncertainty bounds maps for Volcán de Colima, based on 16-17 April, 1991 event.

References

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