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Integration of stochastic models for long-term eruption forecasting into a Bayesian event tree scheme: a basis method to estimate the probability of volcanic unrest

Alexander Garcia-Aristizabal · **Jacopo Selva** ·
Eisuke Fujita

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Abstract Eruption forecasting refers, in general, to the assessment of the occurrence probability of a given eruptive event, whereas volcanic hazards are normally associated with the analysis of superficial and evident phenomena that usually accompany eruptions (e.g., lava, pyroclastic flows, tephra fall, lahars, etc.). Nevertheless, several hazards of volcanic origin may occur in non eruptive phases during unrest episodes. Among others, remarkable examples are gas emissions, phreatic explosions, ground deformation, and seismic swarms. Many of such events may lead to significant damages and for this reason the 'risk' associated to unrest episodes could not be negligible with respect to eruption-related phenomena. Our main objective in this paper is to provide a quantitative framework to calculate probabilities of volcanic unrest. The mathematical framework proposed is based on the integration of stochastic models based on the analysis of eruption occurrence catalogs into a Bayesian event tree scheme for eruption forecasting and volcanic hazard assessment. Indeed, such models are based on long-term eruption catalogs and in many cases allow a more consistent analysis of long-term temporal modulations of volcanic activity. The main result of this approach is twofold: first, it allows to make inferences about the probability of volcanic unrest; second, it allows to project the results of stochastic modeling of the eruptive history of a volcano toward the probabilistic assessment of volcanic hazards. To illustrate the performance of the proposed approach, we apply it to determine probabilities of unrest at Miyakejima volcano, Japan.

A. Garcia-Aristizabal
Center for the Analysis and Monitoring of Environmental Risk (AMRA), Via Nuova Agnano 11, 80125 Napoli, Italy
Tel.: +39 081 7685125
Fax: +39 081 7685144
E-mail: alexander.garcia@amracenter.com

J. Selva
Istituto Nazionale di Geofisica e Vulcanologia, via Donato Creti 12, 40128 Bologna, Italy

E. Fujita
National Research Institute for Earth Science and Disaster Prevention, Tennodai 3-1, Tsukuba, Ibaraki, 305-0006, Japan

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1 Introduction

Probabilistic Volcanic Hazard Assessments (PVHA) is probably one of the most challenging fields of modern volcanology. Volcanic hazard assessment is presented in many different ways, ranging from maps of past deposits of the volcano, to more quantitative probabilistic assessments (e.g., Scandone et al 1993; Newhall and Hoblitt 2002; Marzocchi et al 2004; Martin et al 2004; Neri et al 2008; Marzocchi et al 2008). The latter, being quantitative, has the important advantage of representing the basic component for rationale decision making (e.g., Marzocchi and Woo 2007, 2009; Woo 2008); however, full PVHA applications are still quite rare (e.g., Magill et al 2006; Ho et al 2006; Neri et al 2008; Marzocchi et al 2008; Selva et al 2010).

Volcanic hazards are usually associated with the occurrence of volcanic eruptions; nevertheless, several hazards of volcanic origin may occur in non eruptive phases. Among others, remarkable examples are gas emissions, phreatic explosions, ground deformation and seismic activity (e.g., Newhall and Dzurisin 1988). Such events may lead to significant damages and even push decision makers to take mitigation actions as, for example, the evacuation of population (e.g., Barberi et al 1984).

An anomalous activity in a volcanic area is generically defined as an “unrest”. Providing a definition of unrest is necessarily subjective and can be object of divergence among specialists. Nevertheless, an “operative” definition is generally used for practical reasons (e.g., Selva et al 2012), associating to unrest episodes those periods in which some parameters (observed or measured) show values overcoming a given level of activity subjectively considered as “normal” background activity. While unrest episodes do not necessarily lead to eruptions, volcanic eruptions are always preceded and accompanied by unrest activity (Newhall and Dzurisin 1988). This implies that the probability to have an unrest is greater than the probability of eruption, and if we consider that the non-eruptive volcanic hazards can occur during unrest episodes, it makes the risk associated to unrest episodes not negligible with respect to eruption-related phenomena.

As a consequence, a complete PVHA should also include a probabilistic long-term analysis of the occurrence of unrest episodes. However, this goal implies several issues to be dealt with: first, a quantitative definition of unrest is required, which is, as discussed before, a rather subjective matter. Second, more critical, few past data (i.e., unrest catalogs) are generally available, since a rather developed monitoring system is generally necessary to record the unrest occurrence (Newhall and Dzurisin 1988). Consequently, most of probabilistic analyses concentrate on volcanic eruptions only, while probabilistic inference about unrest is often avoided.

In this paper, we drive the problem of determining unrest probabilities trying to merge the results of analysis of both, eruptive history and unrest catalog data. The eruptive history of a volcano can be used to define stochastic models as mathematical structures to describe the response of a volcanic system; this approach is generally used for long-term eruption forecasting. On the other hand, the Bayesian event tree model (BET, Marzocchi et al 2004, 2008) is a quantitative tool to calculate probabilities related to eruption forecasting (EF) and volcanic hazard (VH) assessments; the EF part in the BET model is mainly based on the analysis of unrest catalogs. These two sources of information can be complementary, and the main objective of this paper is to provide a quantitative framework to integrate stochastic models based on eruptive sequences analysis, into a BET scheme. This operation may be a possible approach to update and estimate probabilities for the occurrence of unrest episodes.

In addition, trough the BET model, the analysis can be extended to quantitatively assess probabilistic volcanic hazards (Marzocchi et al 2010; Selva et al 2010). Therefore, the integration of the information provided by stochastic models into the BET framework may also allow the forward extension of such models for full long-term probabilistic volcanic hazard assessments (PVHA).

2 Proposed methodology

In this section we describe the methodological approach whose main result is the quantification of the probabilities of volcanic unrest. The basic idea is to integrate information provided by two different approaches used for eruption forecasting: (1) the general framework offered by the BET (Marzocchi et al 2004, 2008), which is based on the analysis of unrest catalogs, and (2) the information provided by stochastic models based on the analysis of eruptive sequences. This section is organized as follows. Firstly in section 2.1 we describe the general structure of BET. Then in section 2.2 we illustrate the stochastic models often used in literature to describe eruptive behavior of volcanoes. Finally, section 2.3 contains the core of the paper where the method to integrate the results of stochastic models into the BET structure is proposed.

2.1 The Bayesian event tree model for eruption forecasting

BET is a probabilistic structure for long- and short-term EF and VH based on a fully probabilistic Bayesian scheme (e.g., Marzocchi et al 2004, 2008). It uses a branching scheme in which individual branches are alternative steps from a general prior event, state, or condition, and which evolve into increasingly specific subsequent events (intermediate outcomes) up to a final outcome. The points on the scheme where new branches are created are referred to as nodes (Newhall and Hoblitt 2002; Marzocchi et al 2004, 2008). For the first five nodes of the event tree we have the following states (Fig. 1): **Node 1** (*Occurrence*): unrest/no unrest in the time interval $(t_0, t_0 + \tau)$, where t_0 is the present time and τ is the forecast time window considered; **Node 2** (*Origin*): the unrest is due to *magma* or *other causes* (e.g. hydrothermal, tectonics, etc.), given that an unrest has been detected; **Node 3** (*Outcome*): the magma will reach the surface (i.e. the volcano will erupt), or not, given that the unrest has a magmatic origin; **Node 4** (*Location*): the eruption will occur in a specific location (e.g. crater, a flank, etc.), given that there is an eruption; and **Node 5** (*Magnitude*): the eruption will be of a certain magnitude/size (e.g. VEI), given that there is an eruption in a certain location.

The temporal analysis is fully controlled by Node 1, since the other nodes are usually assumed time independent; as a consequence, temporal aspects rely essentially in the analysis of unrest catalogs.

[FIG. 1 SOMEWHERE AROUND HERE]

At each of these nodes a probability density function (PDF) is assigned; the use of these probability functions (characteristic of a Bayesian approach) allows BET to estimate aleatory (stochastic) and epistemic (data- or knowledge-limited) uncertainties. θ_k^j is the probability of the conditional event j at the k-th node, conditioned to the occurrence of the previous nodes. For each node we have: $[\theta_1^{(unrest)}]$, $[\theta_2^{(magma|unrest)}]$, $[\theta_3^{(eruption|magma,unrest)}]$, $[\theta_4^{(location|eruption,magma,unrest)}]$, and so on, where the square brackets stand for a generic PDF representing epistemic uncertainty on the assessed values. Following this structure, the probability of eruption in particular is calculated using the first three nodes (Eq. 1):

$$[P(\text{eruption})] = [\theta_1^{(unrest)}][\theta_2^{(magma|unrest)}][\theta_3^{(eruption|magma,unrest)}] \tag{1}$$

where the superscripts may be omitted, to simplify notation.

To assess volcanic hazards, further nodes (6 to 8) can be added, in order to assess exceedance probabilities of predefined threshold values for (potentially) all volcanic phenomena of interest (as for example ash fall, pyroclastic and lava flows, or lahars, Marzocchi et al 2008). The treatment of conditional probabilities at each one of these further nodes is completely in agreement with the one for nodes 1 to 5, presented above.

2.2 Stochastic models for long-term eruption forecasting

A time series of eruptions from a single volcano can be treated as a stochastic point process with individual eruptions as (random) independent events in time. Statistical analysis of repose time catalogs have been performed for a large number of volcanoes (e.g., Wickman 1976; Klein 1982; Mulargia et al 1985, 1987; De la Cruz-Reyna 1991; Burt et al 1994; Marzocchi and Zaccarelli 2006; Garcia-Aristizabal et al 2012), mainly where detailed catalogs exist. The main objective of this kind of analysis is to develop probabilistic models to understand the past eruptive activity of the volcano and to forecast its future behavior.

Up to now distinct probabilistic models have been proposed to describe the eruptive behavior of different volcanoes around the world. The most frequent solutions describe the eruptive activity in terms of, e.g., a homogeneous Poisson processes in time domain (e.g., Klein 1982; De la Cruz-Reyna 1991; Marzocchi and Zaccarelli 2006), or a non-homogeneous Poisson process modeled using a Weibull process (e.g., Ho 1991; Bebbington and Lai 1996a,b), or based on the properties of the Brownian passage-time model (Garcia-Aristizabal et al 2012). As consequence of these kinds of modeling, time-dependent, or time-independent EF assessment exclusively based on the eruptive history of the volcano can be performed. Such stochastic models are generally used to estimate eruption probabilities; nevertheless, in very few cases the forecasting is accompanied by an uncertainty quantification, which is a fundamental requirement for the methodological approach proposed in this paper. We argue that independently of its use here, it would be a good practice to propagate the 'known' uncertainties up to the forecasting derived form stochastic models, as for example using bootstrap procedures, or considering the uncertainties from the parameter(s) estimation.

2.3 Integration of stochastic models as 'theoretical beliefs' into a BET scheme: inference for the probability of unrest

The mathematical approaches described in sections 2.1 and 2.2 are two tools to perform EF assessment in a fully probabilistic way. While the BET structure has been designed to be based on the analysis of unrest catalogs, the stochastic models in section 2.2 are based on the analysis of eruptive sequences. Our objective in this section is to describe a method to integrate the EF information provided by both, the 'stochastic' and the BET models; this operation may be fundamental in many applications since (1) the integration of 'stochastic' models in BET would allow to make inference for the parameters of the probability distribution that characterizes the probability of volcanic unrest ($[\theta_1]$ at node 1), (2) it will make compatible forecasting using both approaches, and (3) integrated in BET, this approach allows to project forward the stochastic models toward probabilistic hazard assessment through the event tree scheme of BET. Note that in this paper we deal with long-term EF, then the PDF distributions implicitly refer to the non-monitoring part in BET (see Marzocchi et al 2008).

The temporal analysis in BET is implicit at node 1 (probability of unrest) and it is defined through a static setting of its parameters, i.e., the average and the equivalent number of data of the prior model (Θ_1 and Λ_1 respectively), and past data $y_i^{(1)}$ (for details see Appendix A, and the electronic supplementary material in Marzocchi et al 2008). Once the forecasting window (τ) has been selected, the catalog of unrest events is analyzed and the parameters of BET at node 1 are defined; at this point the temporal constrains are set and do not change through time. In general terms, the EF in BET model is neither completely time-dependent nor completely time-independent; in fact, the statistical assessment simply refers to the next time window, and both prior model(s) and past data may change through time, making consequently change also the BET statistical assessments. In this way, time-dependent analysis may be represented with BET through a series of repeated static analysis whose parameters change through time. In practice, the results of a time-dependent model assessing the probability of a specific event may enter into the BET model through a continuous update of the BET parameters.

In those cases in which databases with the information of past unrest episodes are scarce or considered not sufficiently representative, long-term EF is more likely based on the analysis of past eruptive activity by stochastic modeling (as described in section 2.2), for which usually rather long catalogs are available. A possible output variable of such models may be the absolute probability of eruption θ_E in a given time interval τ'. Our objective is to write the parameters of the BET model in order to fit the values of θ_E as obtained by a given stochastic long-term EF model (that hereinafter we call $\theta_E^{(model)}$). This allows to include the information provided by a given model into the more general framework of EF provided by BET. Assuming that the rate of magmatic unrest episodes respect to the total number of unrest events, as well as the rate of eruptive events respect to the number of magmatic unrest is stationary, then the target of the analysis is directly associated to the update of the probability of unrest.

Note that the eruption forecasting using stochastic models can be performed referred to any time interval τ'. When the distribution representing the probability of unrest is 'updated' using this information, the forecasting in BET will be also performed respect to τ', since the temporal information initially derived from the unrest catalog is lost, and the new forecasting window is τ' (this issue is further discussed in section 3.4).

Referring to BET symbols (for details see Marzocchi et al (2008)), the distribution relative to the absolute probability of eruption in the next time window τ may be defined as:

$$[\theta_E] = [\theta_3]\,[\theta_2]\,[\theta_1] \tag{2}$$

In Eq. 2 the notation in superscripts has been omitted for simplicity, but it should be kept in mind that those are conditional probabilities (as in Eq. 1); then, $[\theta_1]$ represents the probability of an unrest episode in τ, $[\theta_2]$ the probability of a magmatic unrest, given an unrest episode, and $[\theta_3]$ the probability of eruption, given a magmatic unrest.

Therefore, the absolute probability of eruption simultaneously depends on the probability distributions at nodes 1, 2 and 3. In principle, the $[\theta_2]$ and $[\theta_3]$ distributions can be assumed valid only for the time window τ, but their parameters are usually set using all past data (Marzocchi et al 2004, 2008), and for this reason it can be assumed that they do not vary significantly through time. This means that the relative proportion of eruptions from magmatic unrest (node 3) and the proportion of magmatic unrest from generic unrest (node 2) is considered constant over time, and based on time independent considerations. Here, we follow this idea assuming the distributions $[\theta_2]$ and $[\theta_3]$ as constant through time (and with known parameters). Therefore, the BET parameters to be set from $\theta_E^{(model)}$ are the parameters of the posterior $[\theta_1]$ distribution, i.e., its mean Θ_1 and its variance through the equivalent number of data Λ_1.

In general, the stochastic models do not provide the analytic form of $[\theta_E^{(model)}]$ distribution; instead, only the best guess value of $\theta_E^{(model)}$ is evaluated, i.e., $E([\theta_E^{(model)}])$. Sometimes, also other parameters defining $[\theta_E]$ are provided, i.e., its variance or several percentiles, based on statistical analysis and/or bootstrap procedures. Conversely, in the BET model, the statistical distributions $[\theta_1]$, $[\theta_2]$ and $[\theta_3]$ are Beta distributions, while the statistical distribution of θ_E is obtained numerically. This imply that in most of the applications, the analytical form of $[\theta_E]$ will not be known, and the inversion must be done numerically; however, this also implies that Eq. 2 cannot be simply inverted for $[\theta_1]$, since each realization of θ_E strictly depends on specific (and unknown) realizations of θ_2 and θ_3.

The mean of $[\theta_E]$ in BET can be expressed in terms of the means of $[\theta_1]$, $[\theta_2]$ and $[\theta_3]$, i.e., Θ_1, Θ_2 and Θ_3 respectively, being $[\theta_1]$, $[\theta_2]$ and $[\theta_3]$ Beta distributions:

$$E([\theta_E]) = E([\theta_1])E([\theta_2])E([\theta_3]) = \Theta_1\Theta_2\Theta_3 \tag{3}$$

where Θ_2 and Θ_3 are known. Therefore, we can set Θ_1 using our best guess value of $E([\theta_E])$, that is $E([\theta_E^{(model)}])$. In practice, we set:

$$\Theta_1 = \frac{E([\theta_E^{(model)}])}{\Theta_2\Theta_3} \tag{4}$$

Since Θ_1 must be smaller than 1, we have that

$$\Theta_2\Theta_3 > E([\theta_E^{(model)})]) \tag{5}$$

that is always true for well defined assessments, since the probability that an unrest episode leads to an eruption ($\Theta_2\Theta_3$) must be greater than the absolute probability of eruption.

To set the other free parameter, the equivalent number of data at node 1 (Λ_1), we must do several considerations. First, we must have an explicit estimate on the variance of $[\theta_E^{(model)}]$ (or, equivalently, a confidence interval through percentiles), which in most of the applications is not evaluated. If the variance is not quantified, then Λ_1 may be set according to the subjective evaluation of the reliability of the model. Second, when the variance is estimated, we must consider whether such variance correctly represents the entire epistemic uncertainty related to the physical process.

If the variance of $[\theta_E^{(model)}]$ is provided and it is assumed to be representative of the entire epistemic uncertainty, Λ_1 can be set using this information. Unfortunately, an analytic relationship like the one for averages (Eq. 3) does not exists, therefore, Λ_1 must be inferred using an iterative procedure, that is, evaluate $[\theta_E]$ for different values of Λ_1 (e.g., forward modeling with BET) and compare its variance (or percentiles) with the variance (or percentiles) of $[\theta_E^{(model)}]$. A possible procedure might be to determine the parameters that minimize the following misfit function (Eq. 6):

$$\delta V_i = \left\| \text{Variance}\left([\theta_E^{(model)}]\right) - \text{Variance}\left([\theta_E]^{(\lambda_i)}\right) \right\| \tag{6}$$

in which $[\theta_E]^{(\lambda_i)}$ is obtained using Eq. 2, where $[\theta_1]$ is the Beta posterior distribution (at node 1) whose parameters are set with Θ_1 from Eq. 4, and $\Lambda_1 = \lambda_i$, for $\{\lambda_i \in \mathbb{R}^+\}$. In this case, the best guess value for Λ_1 will be the value λ_i (i.e. the equivalent number of data) that minimizes δV_i (Eq. 6). With the estimated values of Θ_1 and Λ_1, the parameters of the first three nodes of the BET will be completely defined, and its long-term probability of eruption assessment (referred now to a forecasting time window τ') will be coherent with the forecast provided by stochastic models based on eruptive catalogs. In this way, the information of the stochastic model can be integrated on a BET scheme, and through BET, it is possible to assess the probability of occurrence of any possible path within the event tree. In section 3.3 we present an algorithm illustrating a possible routine to perform the analysis discussed here. Note that after this operation θ_1 is completely based on $\theta_1^{(model)}$; in addition θ_2 and θ_3 have been assumed time independent. Therefore, the temporal information contained in θ_1 will be exactly coincident with the one of the stochastic model.

3 Practical example for Miyakejima volcano, Japan

We use the case of Miyakejima volcano to illustrate the methodology depicted in the previous section. Miyakejima volcano is one of the most active basaltic volcanoes in Japan; its recurrent eruptive behavior has been observed and a detailed quantitative analysis based on its past activity has been conducted in Garcia-Aristizabal et al (2012). In most historical eruptions, basaltic lava and scoria erupted mainly from flank fissures (Tsukui and Suzuki 1998) and most eruptions lasted a short time (a day to a month). The latest eruptive episode started in June 2000 and a caldera formed at the summit; since then, the volcano has been showing some seismic swarms accompanied by important gas emissions for more than 9 years (Nakada et al 2005; Ueda et al 2005; Garcia-Aristizabal et al 2012).

3.1 Assigning probabilities to the first three nodes of BET for Miyakejima

We will now compute the probability distributions for each of the first three nodes of BET for the non monitoring part (then we will only concentrate in the non-monitoring part for long-term EF assessments). The data and information for this analysis has been provided by the National Research Institute for Earth Science and Disaster Prevention (NIED), Japan. The definition of unrest and magmatic unrest episodes has been based on the analysis of seismic, geodetic (GPS), and geochemical information derived from the databases produced by the monitoring activities of NIED. In particular, to define background levels of seismic activity we apply the criteria for quantitative determination of seismicity-rate thresholds (Chapter 4 in Garcia-Aristizabal 2010), which is based on the Generalized Poisson process. We note that in the case of Miyakejima volcano the change from background activity to unrest is quite easily identifiable since during the non-unrest periods the volcano shows practically no activity (at least considering seismicity and deformation parameters). As first step, a forecasting time window has to be selected; given that the time between the start of an unrest and the start of the subsequent eruption in Miyakejima is often very small, the unit of time selected for the forecasting window is 1 month ($\tau = 1$ month).

3.1.1 BET - Node 1:

As prior information for node 1, we assume absolute ignorance; then the prior distribution for node 1 is set as a Beta distribution (Eq. 11 in Appendix A) with parameters $\alpha = 1,\ \beta = 1$ (that emulates a Uniform distribution):

$$[\theta_1]_{prior} = \mathrm{Beta}(\alpha = 1,\ \beta = 1)$$

Based on the analysis of monitoring databases from 1997, we consider that in the period 1997 - 2011 there has been 1 unrest episode starting on June 26, 2000; then we define ($y_1 = 1$), out of a total of 43 time intervals defined ($n_1 = 43$); the likelihood is then defined as (Eq. 14 in Appendix A):

$$[y_1|\theta_1] = \mathrm{Bin}(1,\ 43;\ \theta_1)$$

the posterior distribution for node 1 is then defined as (Eq. 15 in Appendix A):

$$[\theta_1]_{\text{posterior}} = \text{Beta}\,(2,43) \tag{7}$$

with mean: $\Theta_1^{\text{post}} = 4.4 \times 10^{-2}$.

3.1.2 BET - Node 2:

As prior information for node 2, we again assume absolute ignorance; then the prior distribution for node 2 is set as a Beta distribution (Eq. 11 in Appendix A) with parameters:

$$[\theta_2]_{prior} = \text{Beta}(\alpha = 1,\ \beta = 1)$$

To build the likelihood, we can state that the only mentioned unrest episode had a magmatic origin (since it finished in a magmatic eruption); then, at this node we set: $y_2 = 1$, and $n_2 = 1$; the likelihood is then defined as (Eq. 14 in Appendix A):

$$[y_2|\theta_2] = \text{Bin}(1,\ 1;\ \theta_2)$$

and the posterior distribution at node 2 is defined by (Eq. 15 in Appendix A):

$$[\theta_2]_{\text{posterior}} = [\theta_2 \mid y_2, y_1] = \text{Beta}(2,\ 1) \tag{8}$$

with mean: $\Theta_2^{\text{post}} = 6.7 \times 10^{-1}$.

3.1.3 BET - Node 3:

As prior information, we use the model of Newhall and Dzurisin (1988) for the case of mafic calderas; following this model, about 54% of the well-monitored mafic calderas which enter in unrest have had an eruption. Using this information, we can set that:
$p[\theta_3|\theta_1] = 0.54$;
this relationship can be rewritten as:
$p[\theta_3|\theta_1] = p[\theta_3|\theta_2] \cdot p[\theta_2|\theta_1]$;
from where:
$p[\theta_3|\theta_2] \equiv 0.54/\Theta_2 = 0.81$
Therefore, we can build the prior distribution for this node calculating the parameters α and β of a Beta distribution solving the next system (assuming maximum variance allowed):

$$\begin{cases} \frac{\alpha}{\alpha+\beta} = 0.81 \\ \alpha + \beta = 2 \end{cases}$$

from where we get $\alpha_{prior} = 1.61$ and $\beta_{prior} = 0.39$; the prior distribution at node 3 is then defined as (Eq. 11 in Appendix A):

$$[\theta_3]_{prior} = \text{Beta}(\alpha = 1.61,\ \beta = 0.39)$$

In node 2 we did state that during the period of observation there has been 1 unrest of magmatic origin at Miyakejima, and it finished in eruption; then, for the likelihood function at this node we can set to 1 the number of observed past eruptions ($y_3 = 1$)

out of 1 observed magmatic unrest ($n_3 = 1$); the likelihood is then defined as (Eq. 14 in Appendix A):

$$[y_3|\theta_3] = \mathrm{Bin}(1,\ 1;\ \theta_3)$$

and the posterior distribution at node 3 is then defined by (Eq. 15 in Appendix A):

$$[\theta_3]_{\mathrm{posterior}} = [\theta_3 \mid y_3, y_2, y_1] = \mathrm{Beta}(2.61,\ 0.39); \tag{9}$$

with mean $\Theta_3^{\mathrm{post}} = 8.7 \times 10^{-1}$.

In this way, all the information needed to set the first three nodes of BET has been collected; Table 1 summarizes the results of this analysis and the information needed for the integration of the stochastic model information.

[TAB. 1 SOMEWHERE AROUND HERE]

3.2 Long-term EF based on the eruptive history of Miyakejima volcano

Based on the analysis of a data set containing the repose periods and volumes of lava and tephra emitted by Miyakejima volcano (using information published by Tsukui and Suzuki (1998), from the Global Volcanism Program catalog (Simkin and Siebert 2002), and considering the information of the last eruption (June 2000) from Nakada et al (2005)), Garcia-Aristizabal et al (2012) analyzed the eruptive history of Miyakejima volcano. They concluded that the two-parameters tested models (i.e., Weibull, Gamma, Lognormal, Loglogistic, and Brownian passage-time) are able to explain the observed data, and show a better fit compared to the exponential distribution. While the former four models have been widely used in volcanological literature, the Brownian passage-time (BPT) model seems to be a new interesting alternative to describe volcanological data (Garcia-Aristizabal et al 2012). The BPT is based upon a simple physical model (the Brownian relaxation oscillator), and is parameterized by the mean rate of event occurrence, μ, and the aperiodicity about the mean, α. The Brownian passage-time family differs from other usual candidate distributions for long-term EF in that it may be more readily interpreted in terms of the volcanic processes (Garcia-Aristizabal et al 2012).

To perform our test, we consider the results presented by Garcia-Aristizabal et al (2012) for Miyakejima volcano; following this work, we consider that the BPT model with parameters $\mu = 44.2(\pm 6.5)$ years and $\alpha = 0.51(\pm 0.01)$ successfully describes the long-term eruptive behavior of the volcano, and use this information to calculate the long-term probabilities of eruption (i.e. $E(\theta_E^{(model)})$) and their respective uncertainties (i.e. σ_E^2). For instance, if we denote T the time to the next eruption, it is possible to calculate the probability $Pr_{(x \leq T \leq x+\tau')}$ that an eruption will happen in a time interval $[x, x+\tau']$, given an interval of $x = (t - t_L)$ years since the occurrence of the previous eruption (where t is the time at which the forecasting is performed, and t_L is the time of the last eruption, in years) using the following expression (Eq. 10, e.g., Bowers et al 1997; Garcia-Aristizabal et al 2012):

$$Pr(x \leq T \leq x+\tau')|T \geq x) = \frac{\int_x^{x+\tau'} f(s)ds}{1 - F(x)} \tag{10}$$

where $f()$ and $F()$ represent, respectively, the Probability Density Function and the Cumulative Distribution of the variable representing the repose times. In this specific case $f()$ and $F()$ are represented by the BPT, but they can be replaced by any distribution describing the data of the particular problem. Note that τ' is then the forecasting time window used for EF considering the stochastic model.

Using Eq. 10, and considering the BPT model parameters (and relative uncertainties) calculated by Garcia-Aristizabal et al (2012), we estimated probabilities of eruption and respective uncertainties for different time periods (τ') in the future. The results of those analyses are summarized in Table 2.

[TAB. 2 SOMEWHERE AROUND HERE]

3.3 General algorithm for parameter estimation

In the previous sections we have calculated the necessary information for the BET setting and for the stochastic EF model; at this point, by combining both sources of information it is possible to perform inference to update the parameters that characterize the posterior distribution that represents the probability of unrest (i.e., posterior distribution at node 1 in BET). A simple algorithm for the parameter estimation may be the next:

1. Define stochastic EF model parameters: $E(\theta_E^{(model)})$, σ_E^2;
2. Define $[\theta_2]$ = Beta(α_2, β_2), and $[\theta_3]$ = Beta(α_3, β_3); (this information may be generated using the BET_EF code (e.g., Marzocchi et al 2008), or the procedure used in the previous section);
3. Estimate Θ_1 using Eq. 4;
4. Iterate: For $\lambda_i \geq 1$ (where $\lambda \in \mathbb{R}^+$)
   (a) Estimate $\alpha_{1,i}$ and $\beta_{1,i}$ of $[\theta_1]$ (Eqs. 12 and 13), using Θ_1 (step 3) and $\Lambda_{1,i} = \lambda_i$;
   (b) Estimate $[\theta_E]^{(\lambda_i)}$ using Eq. 2 (by Monte Carlo simulations or forward modeling using BET), where $[\theta_1] = [\theta_{1,i}]$;
   (c) Estimate mean and variance of $[\theta_E]^{(\lambda_i)}$;
   (d) Calculate the misfit function δV_i (Eq. 6);
5. End of Iterative process;
6. Identify the value of λ_i that minimizes the function $\delta V(\lambda)$ (e.g., in a plot of λ_i vs. δV_i, as in Fig. 2).

Note that theoretically $\lambda \in \mathbb{R}^+$, nevertheless, given the interpretation of λ as the 'equivalent number of data' we generally use integer values. We argue that the induced error is negligible since λ may have a domain of different orders of magnitude.

3.4 Results

Using the information of nodes two and three of BET, the results of the long-term EF produced by a stochastic model (in this case the BPT model of Garcia-Aristizabal et al (2012)), and the algorithm depicted in section 3.3, we can now make inference on model parameters to *update* $[\theta_1]$ (i.e., the posterior distribution representing the probability of unrest). Fig. 2 shows the plot of the misfit function δV_i for the different τ' considered in Table 2: (a) 2012, (b) 2012-2015, (c) 2012-2020, and (d) 2012-2025, where the information that no eruption occurred up to year 2011 is included. From this misfit function, the best values for Λ_1 (i.e., the value that minimizes the misfit) can be estimated for each analyzed case. The results obtained are summarized in Table 3, where we report, for each time interval considered, the inferred values of Λ_1^{post}, as well as the α^{post} and β^{post} parameters of the posterior Beta distribution characterizing the probability of unrest after considering the information provided by the stochastic model. In the last column of table 3, the mean of the 'updated' posterior distribution is also reported. Here we refer as 'updated' distribution to $[\theta_1]$ posterior after considering the information provided by the stochastic model.

[FIG. 2 SOMEWHERE AROUND HERE]

[TAB. 3 SOMEWHERE AROUND HERE]

Fig. 3 shows the PDFs of the 'updated' posterior Beta distributions at node 1, $[\theta_1]$, for the different time periods considered in Table 2: (a) 2012, (b) 2012-2015, (c) 2012-2020, and (d) 2012-2025. The PDF of the posterior Beta distribution at node 1 determined from the unrest catalog, and for $\tau = 1$ month, is shown in Fig. 4. The examples presented in the four plots of Fig. 3 are referred to different forecasting cases considering different time intervals τ', and in all of them the information that no eruption occurred up to 2011 is implicitly included (through Eq. 10). It is worth of note that given the time-dependent characteristics of the stochastic model considered, in this particular example it is also possible to determine probabilities for the same τ' (e.g., 1 year) and assuming the hypothesis that no eruption occurs up to some future time (e.g. in the next 10 or 20 years).

[FIG. 3 SOMEWHERE AROUND HERE]

[FIG. 4 SOMEWHERE AROUND HERE]

The results presented in Figs. 3 and 4 allow to analyze the behavior of the proposed methodology to estimate the probability of volcanic unrest. For instance, the PDF shown in Fig. 4 is the result of the analysis of an unrest catalog as performed following the BET approach, and reflects the direct knowledge about the unrest activity at the volcano; its reliability depends on both the length of the observation time and the representativeness of the number of unrest episodes during the observation time. By construction, it could be updated in the future as time passes widening the observation time window and eventually new unrest episodes are registered. However, its results strictly depend on the length of unrest catalogs, which in many cases can be considered not enough representative for long-term assessments. Conversely, the reliability of the results presented in Fig. 3 depends on different factors, as the quality of the stochastic model considered (and then implicitly the quality of the catalog of eruptions used as input data to define the stochastic model), and the proper setting of nodes 2 and 3 in the Bayesian event tree.

Considering the mean of the distributions (e.g. tables 2 and 3, and Fig. 3), as expected, the probability of unrest monotonically increases as the probability of eruption increases; on the other hand, the 'shape' of the distribution tends to enlarge as τ' increases, reflecting an increasing uncertainty (mainly epistemic). This is also evident from Fig. 2, where the Λ tends to decrease from the case (a) to (d), which is by definition an increase on uncertainty.

Finally, it is worth of note the flexibility of the model to perform forecasting exercises, allowing the definition of any forecasting time window (through τ'). This fact is particularly useful for hazard and risk assessment, for example in a multi-risk perspective, where the harmonization of results is necessary.

4 Discussion and final remarks

In this paper we present a quantitative framework to integrate stochastic models (based on the analysis of eruptive sequences), into the Bayesian event tree (BET) structure. This approach may be extremely helpful for a comprehensive EF assessment in volcanoes in which both catalogs of unrest and past eruptions exist. The most important result of this approach is that it allows to make inference for the probability of volcanic unrest accounting for the usually large information contained in catalogs of eruptions. Furthermore, using this procedure it is possible to project forward the results of the stochastic models toward the higher-level nodes in BET. In this way, as new observations (e.g. unrest data) continuously feed the BET and update forecasting (e.g., at nodes 2 and 3), information from stochastic models also may update the BET parameters (through node 1), improving in this way the probabilistic determination of the occurrence of volcanic unrest episodes. Indeed, this operation allows to use the most determinant information from each source; for example, long-term modulations in the volcanic activity cannot be tracked in short and incomplete unrest catalogs, but they can be analyzed in catalogs of eruption that may cover up to several centuries (Bebbington 2010). In addition to this, the presented procedure allows assessing the unrest probability also for volcanoes in which catalogs of unrest are not available at all: indeed, nodes 2 and 3 of BET may be set according to the general features of analog volcanoes (Marzocchi et al 2004), also accounting for the consequent increase in epistemic uncertainty.

The interest on the quantification of volcanic unrest probabilities has different motivations. First, it is useful in a BET analysis in order to improve probability quantifications in any path in the whole event tree, which ranges from eruption forecasting to volcanic hazard assessment. Second, it is important for a ‘holistic’ volcanic hazard assessment. As discussed before, volcanic hazards are usually associated with superficial phenomena produced during volcanic eruptions, and it explains the main interest on quantifying eruption probabilities. Nevertheless, several phenomena of volcanic origin (e.g., gas emission and phreatic explosions, ground deformation, seismic activity, etc.) may occur during unrest episodes in non-eruptive phases, and such events may lead to damages. An example of that is the case of the unrest at Campi Flegrei (Italy) in 1982-1984, where both deformation and seismic activity produced localized damages to buildings (e.g. Vilardo et al 1991; Troise et al 1997; Orsi et al 1999). Then, the risk associated to unrest episodes is not negligible with respect to eruption-related phenomena, and consequently it supports the importance of assessing volcano unrest probabilities.

It implies however the need of a functional definition of unrest. As discussed in the introductory part, a definition of unrest is subjective, but it may not be easy to find a general consensus about it. Nevertheless, in the volcanic surveillance practice it is often the case that the experience allows to define, quantitatively or empirically, a background level of activity for a given active volcano. This fact defines the roots for the operational definition of unrest, defining as ‘unrest episodes’ those periods in which one or more monitored parameters show anomalous values out of the range considered the normal background. In this way, ‘unrest episodes’ can be considered as ‘events’ and can be subject of interest for probabilistic analysis.

We want to highlight here that in the case of volcanoes in which a more or less complete catalog of past unrest episodes is available, the methodology proposed in this document would be unnecessary, since in that case unrest probabilities could be accurately assessed using directly the unrest catalog. Furthermore, if a detailed catalog of past eruptions is also available, then it could be a case to validate the results provided by the model presented here. In fact, validation of probabilistic models is a frequent concern on practical applications, where we want answer the question: is the model a good representation of reality?

To answer this question, three facets of the model-building process could be assessed, i.e., validation of model inputs, model outputs, and of modeling process. Validation of model inputs is mandatory when gathering the data, performing completeness analysis, analyzing the correctness of historical data in catalogs, etc. Good historical data can be also useful to validate model outputs; as described in the previous paragraph, in this case we would require accurate catalogs of unrest episodes and of past eruptions to assess the representativeness of the output of the model (i.e. the updated distribution at node 1) respect to that obtained analyzing a catalog of unrest episodes. It requires to hold back some data and not use it in the model-building process in order to use it to validate the model output. Nevertheless, the fact of analyzing ‘rare events’ is the main problem of testing the performance of probabilistic models in many volcanological applications (as in long-term assessments in other geoscience fields). Consequently, the frequent case of having few data for confidently building a model, makes it impossible to hold back some data for the validation. Then, without the possibility to validate model outputs, we have few options but to try to validate the ‘reasonableness’ of the process. This can be achieved, for example, analyzing the logic behind the model-building process and asking experts to examine the model and/or its results to determine if they are reasonable, and this is the main tool that we have here to validate both modeling process and outputs.

As example to illustrate the performance of the methodology, we analyzed data from Miyakejima volcano, Japan. The first three nodes of BET were set using databases and information provided by NIED; on the other hand, a stochastic model (BPT) based on the eruptive history of Miyakejima volcano was used to produce long-term eruption probabilities for different time periods. Using this information and the methodology presented in this paper, the probability of unrest at Miyakejima volcano was estimated for different time periods through the estimation of the parameters of the posterior distribution of the node 1 of BET, which is a Beta distribution with α and β parameters as described in Table 3; the mean of that distribution, Θ_1, which can be understood as our ‘best guess’ of the probability to have an unrest at Miyakejima volcano are 1.62×10^{-2} for 2012, 5.17×10^{-2} for the period $2012-2015$, 1.70×10^{-1} for the period $2012-2020$, and 3.49×10^{-1} for the period $2012-2025$. As expected, the probability of unrest monotonically increases as the probability of eruption increases, whereas uncertainty also increases as the time period of the forecasting increases.

Further its natural utility for volcano monitoring, this procedure may be useful for short-term hazard and risk assessment, since propagating this information through the Bayesian event tree, it is possible to update practically in real time hazard and risk quantifications. Furthermore, the flexibility of the model to perform forecasting allowing the definition of any forecasting time window (through τ'), simplifies the integration of volcanic hazard and risk assessment results in a multi-risk perspective where harmonizations is required.

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Table 1 Model parameters for the nodes 1, 2, and 3 defined in BET for the Miyakejima volcano example.

Node (k)	$[\theta_k]^{\text{posterior}}$	Mean (Θ_k)
1	$\text{Beta}(2,43)$	4.4×10^{-2}
2	$\mathbf{Beta(2,1)}$	$\mathbf{6.7\times10^{-1}}$
3	$\mathbf{Beta(2.61,0.39)}$	$\mathbf{8.7\times10^{-1}}$

Table 2 Probability of eruption of Miyakejima volcano for different time periods using the BPT model.

Time period	τ' (years)	$E(\theta_E^{(model)})$	σ^2
2012	1	9.4×10^{-3}	4.48×10^{-4}
$2012-2015$	4	3.0×10^{-2}	5.62×10^{-3}
$2012-2020$	9	9.9×10^{-2}	3.03×10^{-2}
$2012-2025$	14	2.0×10^{-1}	6.97×10^{-2}

Table 3 Parameters of the posterior distribution representing the probability of unrest, $[\theta_1]$, for the different periods at which the forecasting has been performed; Λ is the value that minimizes the misfit function presented in Eq. 6; α and β are the two parameter that characterize the posterior Beta distribution at node 1, and Θ_1 the mean of that distribution.

Posterior at Node 1: $[\theta_1]^{\text{post}}=\text{Beta}(\alpha^{\text{post}},\beta^{\text{post}})$				
Time period	Λ_1^{post}	α^{post}	β^{post}	Θ_1
2012	62	2.10×10^{-2}	19.98	1.62×10^{-2}
$2012-2015$	14	7.76×10^{-1}	14.22	5.17×10^{-2}
$2012-2020$	7	13.6×10^{-1}	6.64	1.70×10^{-1}
$2012-2025$	7	27.9×10^{-1}	5.21	3.49×10^{-1}

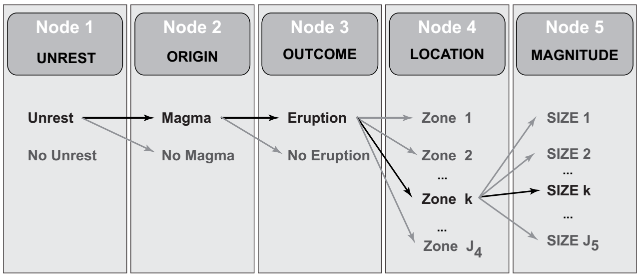

Fig. 1 The first five nodes of the Bayesian event tree (e.g., Marzocchi et al 2004, 2008).

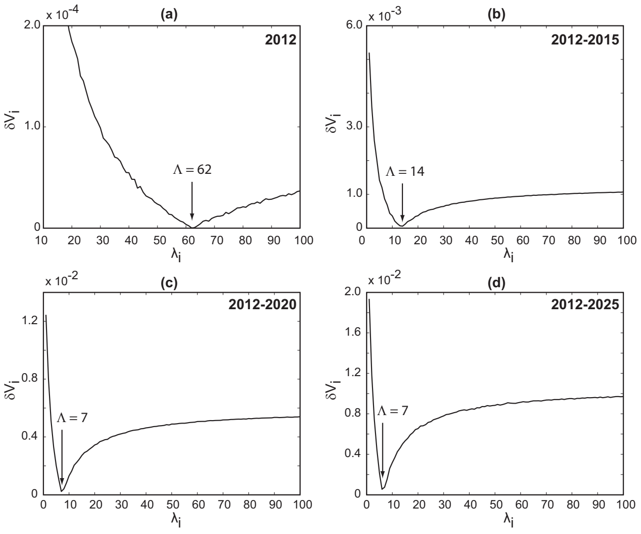

Fig. 2 Plot of λ_i vs. δV_i. The minimum of this function indicates the best value for the Λ_1 model parameter. Inferences for different time periods have been performed: (a) 2012, (b) 2012-2015, (c) 2012-2020, and (d) 2012-2025. The time-dependent characteristics of the BPT model have been considered.

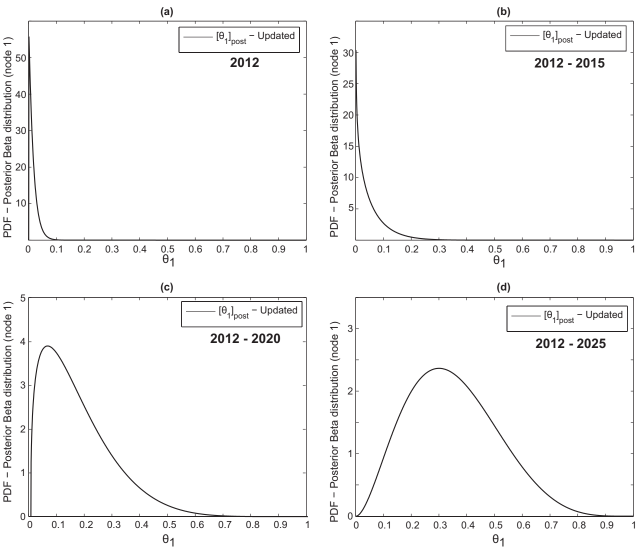

Fig. 3 Plot of the PDF of the posterior Beta distribution at node 1 after the update procedure using the information of a stochastic model. Inferences for different time periods have been performed: (a) 2012, (b) 2012-2015, (c) 2012-2020, and (d) 2012-2025. The time-dependent characteristics of the BPT model have been considered.

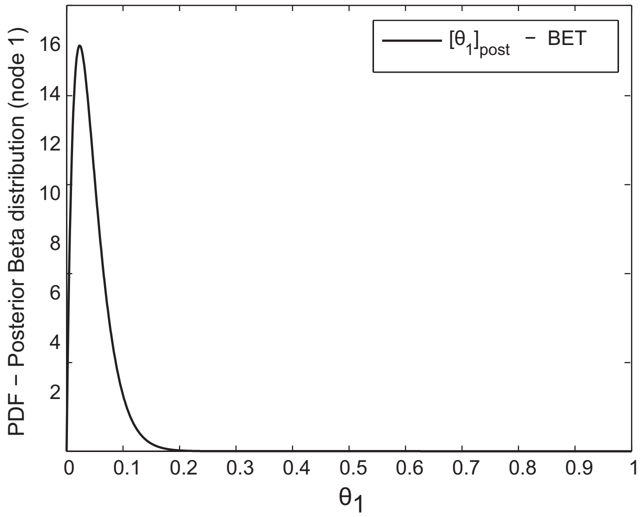

Fig. 4 Plot of the PDF of the posterior Beta distribution at node 1 as initially obtained in BET from unrest data.

A Posterior distribution for $[\theta_k]$ at node k

Here we report the background calculations as defined in the Bayesian event tree model for eruption forecasting (BET_EF) for the k^{th} node, to estimate the posterior distribution, $[\theta_k]$ for $k = 1,2,3$. More detailed descriptions may be found in the supplementary material in Marzocchi et al (2008).
The prior distribution for θ_k is defined as the Beta distribution:

$$[\theta_k]^{prior} = \text{Beta}\,(\alpha_1^{prior}, \beta_1^{prior}) \tag{11}$$

where parameters α_k and β_k (for a generic Beta distribution) are determined by

$$\alpha_k = \Theta_k\,(\Lambda_k + 1) \tag{12}$$

$$\beta_k = (\Lambda_k + 1) - \alpha_k \tag{13}$$

where Θ_k is the central value (i.e. the mean) inferred by *a priori* information (models, theoretical beliefs, etc.), and Λ_k is the so called *equivalent number of data* (for details see Marzocchi et al 2008).
For the *likelihood* function at node k, the two possible outcomes can be treated as *success* (e.g. eruption) and *failure* (e.g.non-eruption) using a binomial model under some specific conditions. y_k is a variable that counts the number of (non-overlapping) time windows which contain a *success* (e.g., for node 1 an onset of unrest), in a set of n_k observations (or trials); using the binomial distribution, the likelihood function is defined as:

$$[y_k|\theta_k] = \text{Bin}(y_k, n_k; \theta_k) \tag{14}$$

The choice of the Beta and Binomial (or Dirichlet and Multinomial in the multivariate case) distributions simplifies the computation because they are conjugate distributions (e.g., Gelman et al 1995). A Dirichlet multiplied by a Multinomial is still a Dirichlet. Then, the *posterior* distribution for $[\theta_k]$ is:

$$[\theta_k] = [\theta_k|y_k] = Beta\,(\alpha_k + y_k, \beta_k + n_k - y_k) \tag{15}$$