Båth's law and the self-similarity of earthquakes

Rodolfo Console, Anna Maria Lombardi, and Maura Murru

Istituto Nazionale di Geofisica e Vulcanologia, Rome, Italy

David Rhoades

Institute of Geological and Nuclear Sciences, Lower Hutt, New Zealand

[1]   We revisit the issue of the so-called Båth's law concerning the difference D_1 between the magnitude of the main shock and the second largest shock in the same sequence. A mathematical formulation of the problem is developed with the only assumption being that all the events belong to the same self-similar set of earthquakes following the Gutenberg–Richter magnitude distribution. This model shows a substantial dependence of D_1 on the magnitude thresholds chosen for the main shocks and the aftershocks and in this way partly explains the large D_1 values reported in the past. Analysis of the New Zealand and Preliminary Determination of Epicenters (PDE) catalogs of shallow earthquakes demonstrates a rough agreement between the average D_1 values predicted by the theoretical model and those observed. Limiting our attention to the average D_1 values, Båth's law does not seem to strongly contradict the Gutenberg–Richter law. Nevertheless, a detailed analysis of the D_1 distribution shows that the Gutenberg–Richter hypothesis with a constant b-value does not fully explain the experimental observations. The theoretical distribution has a larger proportion of low D_1 values and a smaller proportion of high D_1 values than the experimental observations. Thus, Båth's law and the Gutenberg–Richter law cannot be completely reconciled, although based on this analysis the mismatch is not as great as has sometimes been supposed.   *INDEX TERMS:* 7223 Seismology: Seismic hazard assessment and prediction; 7215 Seismology: Earthquake parameters; 7260 Seismology: Theory and modeling; *KEYWORDS:* Båth's law, b-value, self-similarity, Gutenberg-Richter relation, ordered statistics, cluster

1.   Introduction

[2]   The relation between the magnitude of the main shock M_0 in the sequence and its largest aftershock M_1 is still an open problem. Several attempts have been made to determine statistical relations between these magnitudes. In various circumstances the difference $D_1 = M_0 - M_1$ was found approximately equal to the constant value 1.2 (assuming that both magnitudes are reliable). This property appears independent of the absolute magnitudes of the shocks concerned and other aspects of the particular sequence under study. This relation, usually called *Båth's law* in the seismological literature [*Richter*, 1958; *Båth*, 1965], implies that the seismic energy of the main shock is on average about 50 times as large as the energy of the largest aftershocks [*Båth*, 1965]. However, Båth noted possible exceptions to his law, as in the case of a group of several equally large main shocks and aftershocks. The results obtained by *Utsu* [1957, 1961] on Japanese aftershock sequences, though the D_1 parameter was distributed

over a rather large range of values, seem to confirm the above mentioned trend as an average. *Utsu* [1969] interpreted these results as a proof that main shocks belong to a different category from all the other events in the aftershock sequence. Similar results have also been reported in other studies on the statistical distribution of D_1 [*Papazachos*, 1974; *Purcaru*, 1974; *Tsapanos*, 1990]. They agree on average with Båth's law, taking into account differences in the methods used for selection of main shock–aftershock pairs. *Vere-Jones* [1969] discussed a possible different interpretation of Båth's law based on the hypothesis that the magnitudes of the shocks in an aftershock sequence are independently and exponentially distributed according to the usual ''Gutenberg–Richter'' frequency–magnitude law. He assumed that the two shocks, as mentioned above, are just the largest and second largest members of a random self-similar sample (in disagreement with *Utsu* [1969]). According to the mathematical theory, the difference between the largest and next-largest members of a sample randomly chosen (with the same lower limits) from an exponential distribution is independent of the sample size and is exponentially distributed with the same parameter as the distribution of the individual sample

members [*Feller*, 1966, p. 18]. The theoretical model, illustrated by Vere-Jones in more detail in 1975, does not confirm Båth's law in two important points: it predicts both an exponential distribution of D_1 with a mean of the order of 0.5, rather than a distribution closely concentrated about the value 1.2, and a positive correlation, rather than zero or negative, between D_1 and M_0 magnitude. These discrepancies have been ascribed by Vere-Jones to the different magnitude thresholds chosen for the definition of the samples from which the strongest and the second largest magnitudes are taken. It does not seem, to our knowledge, that any paper following on the subject has either substantially supported the interpretation given by Vere-Jones, or given an alternative solution to the problem. For instance, *Tsapanos* [1990], implicitly assuming the validity of Båth's law, points out regional variations in the $\overline{D_1}$ value. More recently, *Guo and Ogata* [1997], in their statistical study of aftershock properties, and *Evison and Rhoades* [2001], treating the predictability of the main shock parameters, quite clearly share the classic *Utsu's* [1969] view, i.e., the strongest shock in a sequence is not a member of the self-similar set of aftershocks.

[3] In this paper, developing the formulation of the conditional distribution for D_1 initially introduced by *Vere-Jones* [1975, p. 816] we show how the observed distributions depend both on the particular cutoff value chosen for the lower limit of the two magnitudes, M_0 and M_1, and on the number of events N in each sample. We find that, in the particular case when the difference between the two cutoff magnitude values is equal to 2 units of magnitude and N is approximately equal to 10, the theoretical density function of the magnitude difference is strongly peaked near 1.2 (as predicted by Båth's law). In the limit case of $N \to \infty$ the distribution D_1 is represented by the negative exponential Gutenberg–Richter's law, regardless of the difference in the magnitude thresholds. It will also be shown that the experimental distribution of D_1 is well fitted by the exponential distribution, independently of N, if the cutoff magnitude values are the same for M_0 and M_1.

2. Mathematical Background

[4] In this paper we adopt the hypothesis that the magnitudes of a set of seismic events, observed in a given region and a given time interval, follow the Gutenberg–Richter law, i.e.,

$$\log_{10}(N(M)) = a - bM \tag{1}$$

where $N(M)$ is the number of events with magnitude larger than or equal to M. Equation (1) is equivalent to the statement that the above mentioned magnitudes represent a sample of independent and identically distributed random variables, with density function

$$f(M) = \beta e^{-\beta(M - M_c)} \quad M \geq M_c \tag{2}$$

where M_c is the completeness threshold of the observed magnitudes and $\beta = b \ln(10)$. The cutoff magnitude M_c for a sequence is the lowest magnitude above which the data set is considered complete. It may be taken as the lower bound

of the interval where the cumulative log frequency curve of the magnitudes follows the linear form predicted by the Gutenberg–Richter relation.

[5] It is known that [*Feller*, 1966; *Casella and Berger*, 1990], given N independent and identically distributed random variables with density function (2), the corresponding order statistics $M_0 \geq \ldots \geq M_{N-1}$ are not independent and M_i has a density function

$$f_{M_i}^N(M) = \frac{N!}{i!(N - i - 1)!} \beta e^{-\beta(M - M_c)} (1 - e^{-\beta(M - M_C)})^{N-i-1}$$

$$\cdot (e^{-\beta(M - M_c)})^i \qquad i = 0, \ldots, N - 1. \tag{3}$$

Furthermore, it follows that, if $N \geq 2$, the random variable $D_1 = M_0 - M_1$ has an exponential distribution with parameter β, independently of N. Therefore, having denoted with $\mathbb{E}[D_1]$ the average of D_1, we have [see, for details, *Vere-Jones*, 1969, equation (2)]:

$$\mathbb{E}[D_1] = \frac{1}{\beta}. \tag{4}$$

Now let M_0^* be a constant larger than or equal to M_c and suppose that we have the further information that

$$M_0 \geq M_0^*. \tag{5}$$

The *density function* of D_1 *conditioned* by (5) is:

$$f_{\{D_1|M_0 \geq M_0^*\}}^N(d)$$

$$= \begin{cases} \dfrac{\beta e^{-\beta d}\left[1 - \left(1 - e^{-\beta\left(M_0^* - M_c\right)}e^{\beta d}\right)^N\right] - \beta N e^{-\beta\left(M_0^* - M_c\right)}\left(1 - e^{-\beta\left(M_0^* - M_c\right)}e^{\beta d}\right)^{N-1}}{1 - \left(1 - e^{-\beta\left(M_0^* - M_c\right)}\right)^N} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } d \leq M_0^* - M_c \\[1em] \dfrac{\beta e^{-\beta d}}{1 - \left(1 - e^{-\beta\left(M_0^* - M_c\right)}\right)^N} \qquad \text{if } d > M_0^* - M_c \end{cases}$$

$$\tag{6}$$

The relative *conditioned average* is:

$$\mathbb{E}^N[D_1|\{M_0 \geq M_0^*\}] = \frac{1}{\beta} + \frac{N e^{-\beta\left(M_0^* - M_c\right)}}{1 - \left(1 - e^{-\beta\left(M_0^* - M_c\right)}\right)^N}$$

$$\cdot \left[M_0^* - M_c - \sum_{k=1}^{N-1} \frac{1 - e^{-\beta(M_0^* - M_c)^k}}{\beta k}\right] \tag{7}$$

From equation (7) is evident that it depends not only on β, but also on the sample size N and on the difference ΔM between the thresholds M_0^* and M_c. In Figure 1 we show examples of density functions of the magnitude difference,
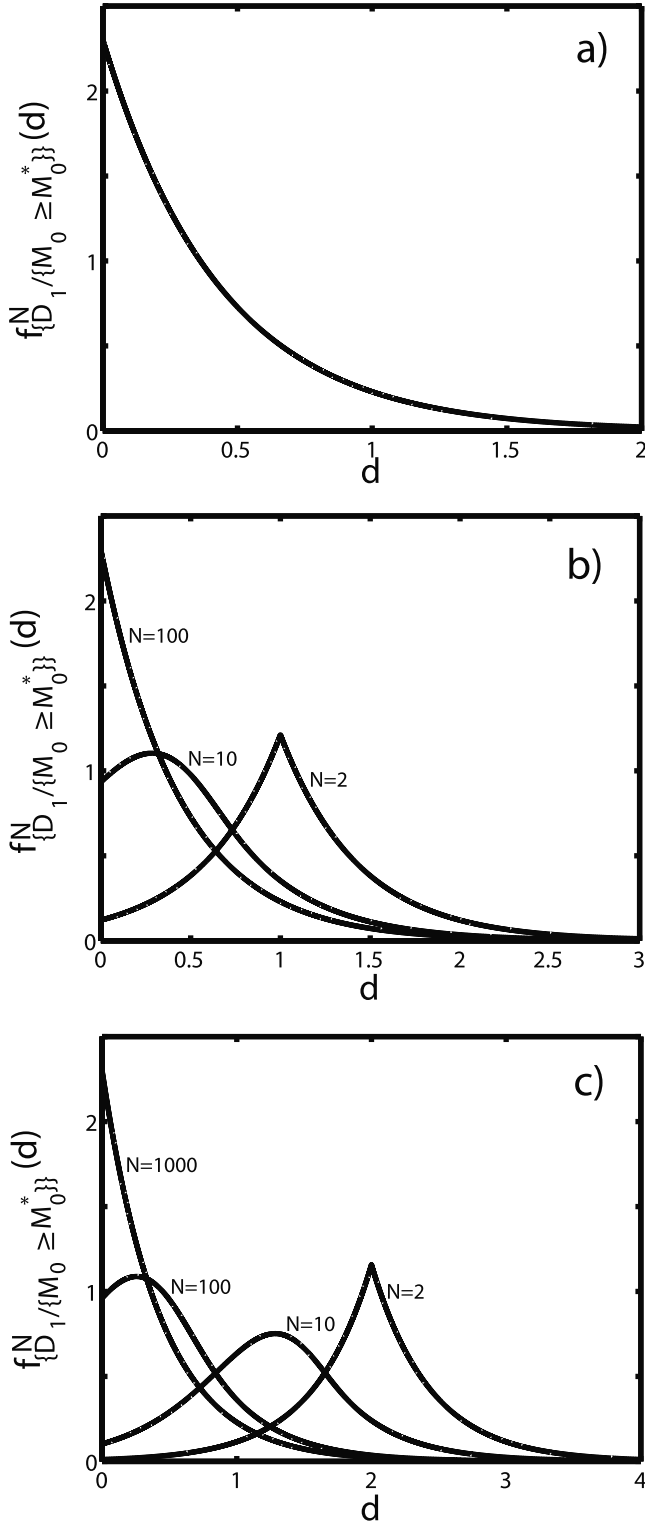
Figure 1. (a) Theoretical density functions of D_1 for the threshold differences $\Delta M = M_0^* - M_c = 0$. The function trend is independent of the sample size. (b) The same as in (a), but for $\Delta M = 1$. Samples for three different sample sizes are shown ($N = 2$, 10, and 100). (c) The same as in (a), but for $\Delta M = 2$. Samples for four different sample sizes are shown ($N = 2$, 10, 100, and 1000).

between the main shock and its largest aftershock, for various ΔM, considering the usual b value equal to 1. The density functions are distinguished on the basis of the sample size. We can observe that, as shown by Figure 1a, if $M_0^* = M_c$ there is no conditioning: the density function of D_1 is coincident with the density function of an exponential random variable for any value of N and, therefore, we have $\mathbb{E}^N[D_1|\{M_0 \geq M_0^*\}] = \frac{1}{\beta}$. Analogously it follows that, for $N \rightarrow +\infty$, the distribution of D_1 converges to an exponential random variable (see Figures 1b and 1c for N equal to 100 and 1000, respectively) and therefore

$$\mathbb{E}\left[D_1|\left\{M^{(N)} \geq M_0^*\right\}\right] \overset{N \rightarrow +\infty}{\longrightarrow} \frac{1}{\beta} \qquad (8)$$

In Figure 2 the trend of $\mathbb{E}^N[D_1|\{M_0 \geq M_0^*\}]$ versus N, compared with the mathematical average of 1000 random variables synthetically obtained through the density function (6), is displayed for $\Delta M = 2$. A more detailed description of the theoretical model and other examples is given by *Lombardi* [2002].

3. Data and Results

[6] This study aims to investigate whether the behavior of real seismicity supports the conclusions of the earlier studies [e.g., *Utsu*, 1961, 1969] regarding the lack of agreement between the Gutenberg–Richter and Båth laws, as far as the main shock magnitudes are concerned. With the aim of testing the statistical model on real data, we used two different data sets. The first is the shallow earthquake ($h \leq 33$ km) catalog compiled by the New Zealand Seismological Observatory, Wellington, for the time span 1 January 1962 to 30 September 1999. We consider all earthquakes contained in a polygonal area bounded by the 165°E and 181°E meridians, and 36°S and 48°S parallels, within which the national network has provided reliable locations for nearly all the events of magnitude (M_l) 4.0 and larger, as
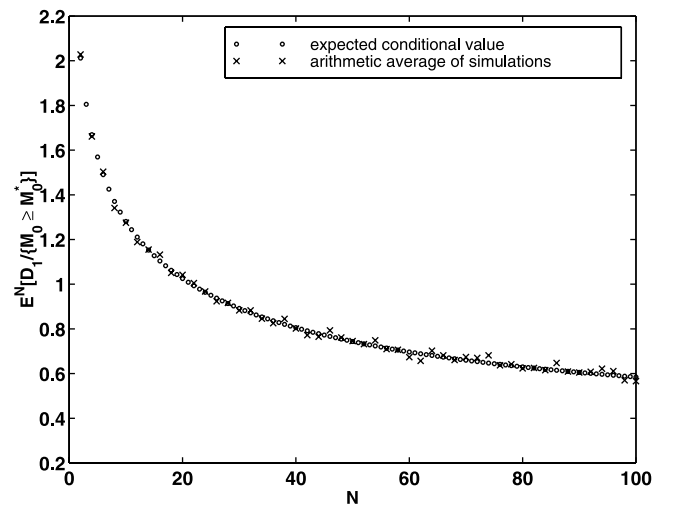

Figure 2. The conditional average of D_1 versus the sample size compared with the arithmetic average of 1000 simulated values for $\Delta M = 2$.
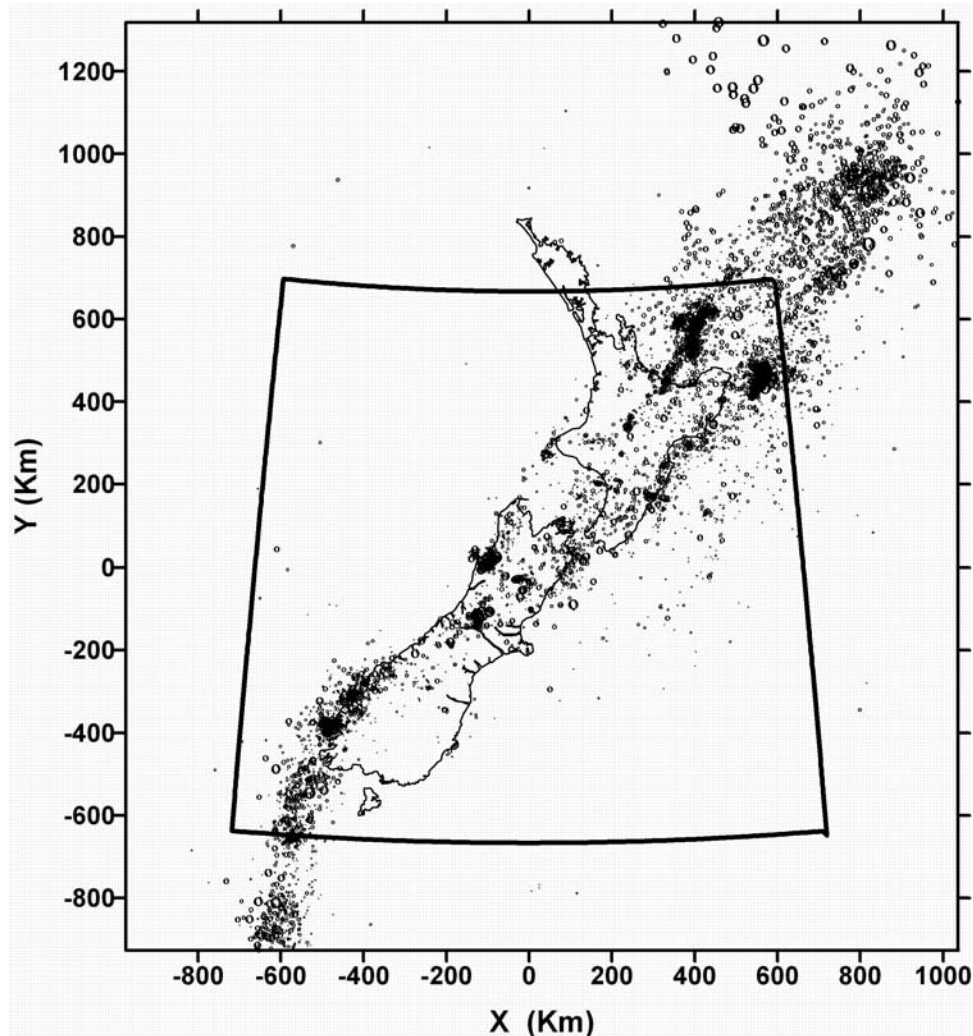
Figure 3.   Epicentral map of the shallow seismicity ($M \geq 4.0$) in the New Zealand region (January 1962 to September 1999). The internal polygon shows the area analyzed in this study. The size of the symbols is scaled in magnitude. The origin of the rectangular coordinates is the point 173°E, 42°S.

shown in Figure 3. The number of events falling in this particular time window and geographical area considered for the analysis is 7,182. This data set is characterized by a b-value of 1.111 ± 0.012, obtained through the maximum likelihood method proposed by *Utsu* [1965, 1967]. The estimate of the standard deviation has been obtained using the equation by *Shi and Bolt* [1982]. It must be noted, however, that a χ^2 test of the magnitude distribution does not fully support the hypothesis of a perfect exponential density function in the lower magnitude range. We decided to adopt 4.0 for the minimum magnitude of the catalog used in this study as a compromise between the size of the data set and its completeness.

[7]   For our purposes we need to identify in the catalog all the subsets of events including a main shock and its aftershocks. Since no standard procedure exists for identification of main shocks and aftershocks, we apply an algorithm requiring a minimum number of subjective definitions. Here aftershocks are defined as events with magnitude M_1 exceeding a threshold M_1^* that are preceded in a given time–space window by another earthquake of

equal or larger magnitude (in this way we consider within the main shock–aftershocks series some sequences that are elsewhere called "multiplets" or "swarms"). In this context the magnitude threshold for aftershocks is considered the same as the completeness magnitude of the whole catalog ($M_1^* = M_c = 4.0$). Main shocks are nonaftershocks with magnitude M_0 exceeding a threshold M_0^* followed by at least one aftershock in the same time–space window mentioned above. All the remaining earthquakes are defined as single events. The values of the time and space parameters needed in this definition are drawn from the algorithm of *Reasenberg* [1985], that takes into account the magnitudes of the preceding main shocks. In particular, for the calculation of the time limits within which we declare an event to belong to a cluster, we use a variant of the original Reasenberg's formula introduced by *Kagan* [1996].

[8]   The analysis has been carried out considering different values of M_0^* (4.0, 5.0, and 6.0, respectively). Table 1 gives the results of our analysis in terms of the number of sequences found in the catalogs and the relative average

Table 1. Statistical Results of the Analysis for D_1 in the New Zealand and in the PDE Catalogs of Shallow Earthquakes

	New Zealand (7182 events)			PDE (29,343 events)		
	$\Delta M = 0$	$\Delta M = 1$	$\Delta M = 2$	$\Delta M = 0$	$\Delta M = 1$	$\Delta M = 2$
N_{cl}	370	124	20	2562	1085	255
N_{ec}	3980	3221	2359	13,177	9430	5938
$\overline{D_1}$	0.4278	0.7984	0.9850	0.5378	0.9214	1.2074
$E[D_1\|M_0 \geq M_0^*]$	0.3910	0.7964	1.0527	0.4241	0.8380	1.2837
σ	0.4041	0.4198	0.4234	0.4948	0.4979	0.5868
$\frac{\sigma}{\sqrt{N_{cl}}}$	0.0210	0.0377	0.0947	0.0098	0.0151	0.0367

N_{cl}: Number of clusters with N events. N_{ec}: Number of nonsingle events. $\overline{D_1}$: Average magnitude difference $\left(\frac{\sum_{i=1}^{N_{cl}} D_1(i)}{N_{cl}}\right)$. σ: Estimated standard deviation $\left(\sqrt{\frac{\sum_{i=1}^{N_{cl}}(D_1(i)-\bar{D}_1)^2}{N_{cl}}}\right)$. $\frac{\sigma}{\sqrt{N_{cl}}}$: Estimated mean square error.

magnitude difference $\overline{D_1}$. It should be noted that most of the clusters, for the New Zealand catalog, have a low number of events. In fact for $M_0^* = M_1^*$, 60% of the clusters have only two events and 80% have a number of events smaller than or equal to 5. The respective values for $M_0^* - M_1^* = 1$ are 40% and 60%.

[9] A visual comparison between the theoretical model and the real seismicity is provided by Figure 4, showing, for the three values of the main shock threshold M_0^*, the histogram of the number of cases observed in each class of D_1. In order to compare the histogram obtained from the observations with the theoretical density function of D_1, $f_{\{D_1|M_0 \geq M_0^*\}}(d)$, we cannot simply use equation (6) because this equation is written for a single value of N. So we make use of the formula

$$f_{\{D_1|M_0 \geq M_0^*\}}(d) = \sum_{N=2}^{+\infty} f^N_{\{D_1|M_0 \geq M_0^*\}}(d) \cdot p_N \qquad (9)$$

where p_N represents the frequency distribution of clusters with N events. In this case, as a theoretical assessment of p_N is not available, it is taken as the observed relative frequency of clusters with N events (if no clusters are observed with N events, p_N is assumed to be 0 and in this case $f^N_{\{D_1|M_0 \geq M_0^*\}}$ does not contribute to the sum). The continuous lines of Figure 4 have been plotted after having applied a proper normalization factor, so that the integral of the density function is equivalent to that of the histograms. The shape of the theoretical curves compared with the relative experimental histograms allows one to judge in a qualitative way how well the model agrees with the observations. In particular, for the case $\Delta M = 2$ (Figure 4c) we must take into account the fact that the number of the main shocks with magnitude equal to or larger than 6.0 is only 20 and that leads to a large dispersion in the observed values.

[10] To obtain the theoretical values of the average magnitude difference $\overline{D_1}$, for the three cases, we compute the weighted average of (7):

$$\mathbb{E}[D_1] = \sum_{N=2}^{+\infty} E^N[D_1|\{M_0 \geq M_0^*\}] \cdot p_N \qquad (10)$$

and so obtain the three values 0.391, 0.796, 1.053 (as shown in Table 1). Both the observed overall values of $\overline{D_1}$ and the corresponding theoretical values show the same tendency to increase with ΔM. Moreover, we may note that the

observed $\overline{D_1}$ values, except for $\Delta M = 0$, are in agreement with the theory within the mean square deviation.

[11] The second data set analyzed in this study is the catalog of Preliminary Determination of Epicenters (PDE) reported by the National Earthquake Information Service (NEIC) from 1 January 1973 to 29 January 2001. The magnitude under analysis is that reported by NEIC in the eighty columns format (the maximum value among M_s, M_b and M_w). The analysis has been limited to events with depth shallower than 50 km and magnitude equal to or larger than 5.0, which seems a suitable completeness threshold for this catalog. The total number of events selected in this way is 29,343. The maximum likelihood value of b for this data set, obtained with the same criteria used for the New Zealand data, is 1.024 ± 0.006.

[12] For the analysis of the PDE catalog we have considered again three different values of M_0^*, but one magnitude unit larger than for the New Zealand case (5.0, 6.0, and 7.0, respectively). The relative results are shown in Table 1. The comparison between the observed overall values of $\overline{D_1}$ and the theoretical ones obtained by applying equation (10) indicates a difference of about 20% for $\Delta M = 0$ and a difference of about 10% for $\Delta M = 1$, the experimental values being larger than the theoretical ones. Unlike what we noted for the New Zealand data, in this case, the range of values defined by the mean square deviation for every ΔM, does not include the expected value of D_1. Furthermore, a visual comparison between the number of cases observed in each class of D_1 and the number predicted by the theoretical model, shown in Figure 5 for the three values of the main shock threshold M_0^*, while still confirming a qualitative agreement, shows some systematic differences. In particular, for $\Delta M = 0$ and $\Delta M = 1$, a deficit of observed cases for $D_1 < 0.6$ and an excess for $D_1 > 1.0$ is quite evident (Figures 5a and 5b). The larger number of earthquakes reported in the PDE catalog allows one to define in a more robust way than with the New Zealand data, the good agreement between the observed and the theoretical trend of the D_1 distribution for $\Delta M = 2$.

4. Discussion

[13] As pointed out by various authors in the past [*Utsu*, 1957, 1961; *Papazachos*, 1974; *Purcaru*, 1974; *Tsapanos*, 1990] and confirmed by the present data analysis, the $D_1 = 1.2$ value predicted by Båth's law must be regarded as the average of a wide range of cases. Some of the same authors studied the D_1 distribution in detail and made various
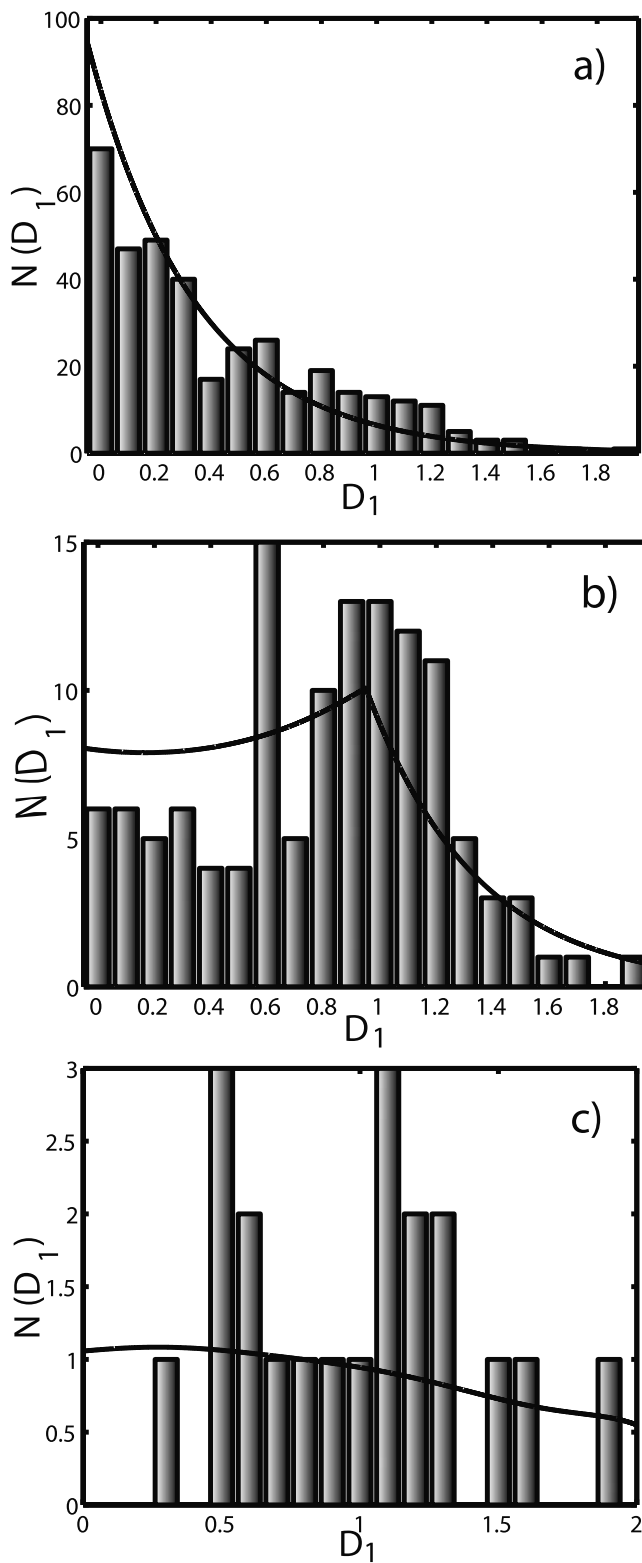
Figure 4. (a) Histogram of the distribution of D_1 observed in New Zealand for the threshold difference $\Delta M = M_0^* - M_1^* = 0$ compared with the theoretical normalized density function. (b) The same as in (a), but for $\Delta M = 1$. (c) The same as in (a), but for $\Delta M = 2$.
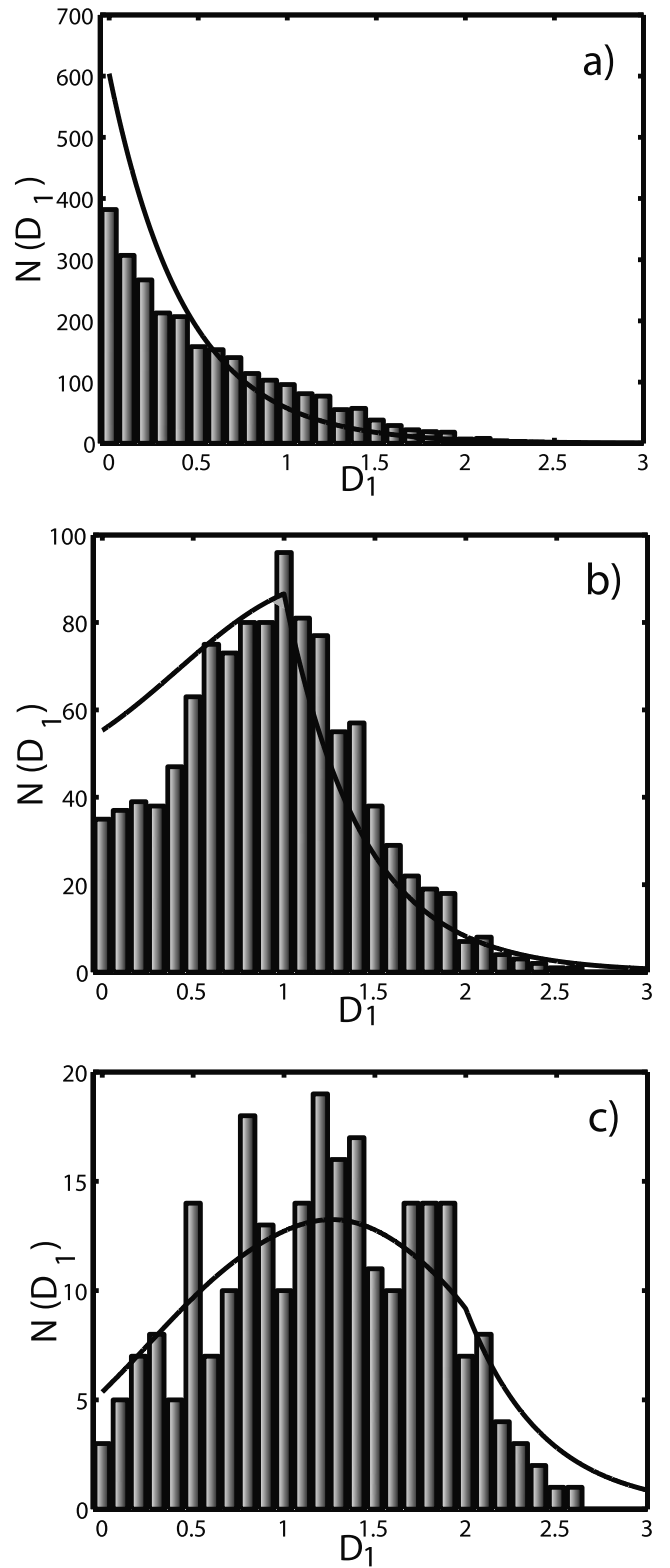
Figure 5. (a) Histogram of the distribution of D_1 observed in the PDE catalog for the threshold difference $\Delta M = M_0^* - M_1^* = 0$ compared with the theoretical normalized density function. (b) The same as in (a), but for $\Delta M = 1$. (c) The same as in (a), but for $\Delta M = 2$.

Table 2. Significance Level of the Theoretical D_1 Distributions Based on the Hypothesis of the Linear Gutenberg–Richter Frequency–Magnitude Relationship

ΔM	New Zealand	PDE
0	<0.001	<0.001
1	0.219	<0.001
2	0.299	0.866

The probabilities are computed for the two catalogs and for three values of the difference ΔM between the thresholds M_0^* and M_1^*.

interpretations. For instance, *Purcaru* [1974] concluded that the hypothesis of D_1 normally distributed, as well as some other different hypotheses including the negative exponential distribution, should be rejected for Greece. *Tsapanos* [1990] observed two distinct peaks (at 1.2 and 1.8 magnitude units) in the D_1 distribution for large circum-Pacific earthquakes and interpreted this circumstance as proof of a different behavior between the convergent plate boundaries and the back arc areas. Though a high significance level of this distinction was claimed by the author, it is difficult to judge because the paper does not report the D_1 distributions observed separately in the two distinct groups of active regions.

[14] In light of the theoretical framework developed in our study, we may surmise that the variety of cases reported in the literature for the D_1 distribution can be ascribed, not only to particular circumstances (such as the b-value or the size of the earthquake cluster), but also to the way in which the data are treated by different authors. Specifically, D_1 depends on the arbitrary choice of the magnitude thresholds M_0^* and M_1^* (typically a difference of $\Delta M = 2$ has been adopted by the aforementioned authors for these thresholds). The $\overline{D_1}$ value can be also biased by the specific definition of main shocks adopted in the analysis. For instance, the arbitrary distinction between main shocks, swarms and multiplets [*Evison*, 1981; *Evison and Rhoades*, 1993] depending on the difference between the magnitude of the largest shock and the third largest shock in the sequence, is expected to have significant influence on the observed distribution of D_1, excluding the smallest values from the average.

[15] In this work we obtain the D_1 distribution with a minimum number of arbitrary assumptions on the definition of main shocks and aftershocks. One of these assumptions is that the magnitude distribution for the two catalogs under study is not subject to the numerous systematic and statistical errors, which frequently affect earthquake detection, earthquake location, and magnitude determination. Main shocks are identified automatically by a computer program based on a quantitative definition, rather than being selected by a subjective inspection of the catalog, as we suspect to be the case in various cases reported in literature. None of the three distributions shown by the histograms of Figures 4 and 5 supports a distinction of the events in separate groups based on the D_1 distribution. The distribution obtained for $\Delta M = 2$ ($M_0^* = 7$ and $M_1^* = 5$) with the world catalog should be close to that reported by *Tsapanos* [1990] but it does not exhibit two distinct maxima at 1.2 and 1.8 magnitude units.

[16] The qualitative agreement between the theoretical model (expressed by the curves of Figures 4 and 5) and the observed data (given by the histograms reported in the same figures) can be better evaluated in a quantitative way by a statistical test. To estimate the significance level of such agreement we used the Monte Carlo method for creating 1000 synthetic frequency distributions (based on our theoretical model of self-similarity in magnitude) for each ΔM case, and compare their likelihood with the likelihood of the respective real data. In doing so, we hypothesize that each value of the synthetic distributions is statistically distributed as a Poisson variable having the mean value equal to the theoretical one. The results of the Monte Carlo analysis lead to the probability that a D_1 distribution randomly obtained under the null hypothesis of the Gutenberg–Richter self-similar magnitude distribution has a likelihood smaller than that of the real distribution. This probability, also called the significance level, is given in Table 2 for the three different values of ΔM and the two catalogs considered in this study. The null Gutenberg–Richter hypothesis should be rejected for $\Delta M=0$ with the seismicity of New Zealand, and for $\Delta M = 0$ and $\Delta M = 1$ with the seismicity of the world.

[17] We considered the hypothesis that the very low significance level obtained for $\Delta M = 0$ with both catalogs is related to the imperfect agreement between the real magnitude distribution of our catalogs and the ideal log linear frequency–magnitude distribution used in the model. In Figure 6 the trend of $log_{10}N(M)$ versus M is shown for both catalogs analyzed. The NZ catalog is characterized by a clear linearity over the whole magnitude range. On the other hand, the plot for the PDE catalog shows a good linearity up to the value 7.8, beyond which the magnitude exhibits a clear saturation This circumstance could affect the D_1 distribution, but we consider it as a minor problem (for $\Delta M = 0$ in particular) because it happens only for less than 100 events on the total data set of nearly 30,000 earthquakes. As a test of the hypothesis, we examined the D_1 distribution obtained by rearranging the events of the real catalog in random order and grouping them in clusters containing the same number of
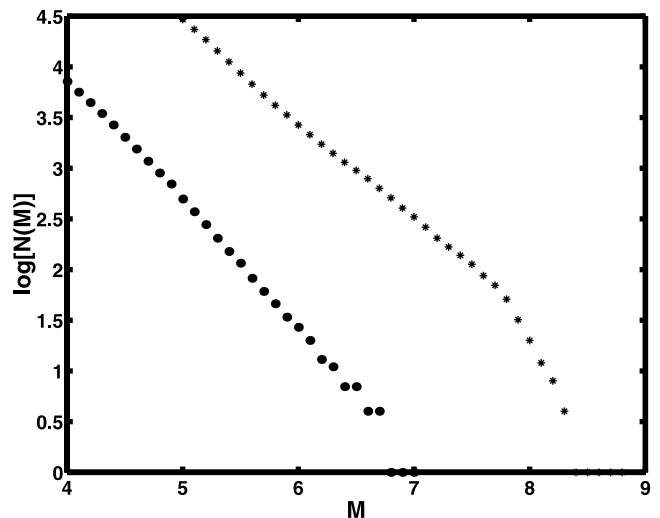

Figure 6. Cumulated frequency-magnitude relation for the New Zealand catalog (shown with dots) compared with the PDE data set (indicated with stars).

events. In this test we used a fixed number of 5 events in each cluster. If the disagreement with the theoretical model depended totally on the magnitude distribution, as the magnitudes used in this test are the same as those present in the real catalog, we would expect to obtain the same significance level as was obtained for the real catalog. The result of the test is that the significance level of the Gutenberg–Richter model is increased greatly by randomizing the events in the clusters. This means that the opposite hypothesis is true, i.e., the magnitude distribution of the earthquakes in natural clusters is significantly different from that assumed in the null hypothesis. To check this idea, we computed the b-value only for the events that, in both catalogs, belong to clusters, including main shocks and their own aftershocks (see Table 1). This gives, for $\Delta M = 0$, $b = 0.996$ and $b = 0.873$ respectively for the New Zealand and the PDE catalog. These values are significantly lower than those obtained from the complete catalogs. By substituting them in equation (7), we obtain $E[D_1] = 0.4360$ for New Zealand and $E[D_1] = 0.4975$ for the PDE, in fairly good agreement with the corresponding observed values of $\overline{D_1}$ reported in Table 1.

[18] It has been shown that the primary events (main shocks and earthquakes with neither aftershocks nor foreshocks) display a teleseismic b-value lower than those reported for the secondary events (aftershocks and foreshocks) and that it could arise simply from the act of choosing main shocks as the largest earthquake in a foreshock–main shock–aftershock sequence [*Knopoff et al.*, 1982; *Frohlich and Davis*, 1993]. From this point of view, these features, that do not depend on the particular earthquake catalog used, whether it is a real or synthetic one, should not be ascribed any physical significance. In this respect, the difference noticed in Figures 4a and 5a and 5b (i.e., the number of large aftershocks is smaller than the prediction based on the Gutenberg–Richter relation) could perhaps even be explained as a consequence of asymmetries in data processing procedures, affecting the b-value and consequently the D_1 distribution.

5. Conclusions

[19] Following the early works of *Vere-Jones* [1969, 1975] we have based our study on a rigorous mathematical formulation of the D_1 distribution with few essential assumptions (all the events follow the Gutenberg–Richter magnitude distribution). These assumptions have proven to be a simple and reliable basis for models of mutual interaction of earthquakes [*Console and Murru*, 2001]. The theoretical distributions obtained in this way are quite similar, with an appropriate selection of the free parameters characterizing the model, with observations carried out and published on this subject for nearly five decades. Through a test carried out on two real and large data sets (the New Zealand and PDE catalogs of shallow earthquakes) we have shown that the hypothesis that the magnitudes of all the earthquakes belong to the same self-similar set of data, can substantially explain the observed $\overline{D_1}$ values without the introduction of any independent rule such as Båth's law. Nevertheless, the objective analysis presented here has also demonstrated a significant difference between the observed and the theoretical D_1 distributions, in that the number of observed D_1 values is smaller than the expected one for $D_1 < 0.6$ and *vice versa* (see Figures 4a and 5a). For the PDE catalog, using the same magnitude thresholds for main shocks and aftershocks ($M_0^* = M_1^* = 5$), the observed $\overline{D_1}$ value is about 20% larger than the theoretical one. Thus, although ignoring the bias introduced by the different magnitude thresholds chosen for main shocks and aftershocks may have been a misleading factor in some past studies, as pointed out by *Vere-Jones* [1969, 1975], the mismatch between Båth's law and the Gutenberg–Richter law [*Utsu*, 1969; *Purcaru*, 1974; *Evison and Rhoades*, 2001] appears to be real. The Gutenberg–Richter law does not entirely explain the observed distribution of D_1 values, unless the b-value computed for the set of earthquakes belonging to the clusters is used in the model. Whether this circumstance is to be interpreted as a change in the physical environment before and after large earthquakes, or rather connected to a bias that has not been completely removed in the statistical process, is a matter that deserves further attention.

References

Båth, M., The energies of seismic body waves and surface waves, in *Contributions in Geophysics in Honor of Beno Gutenberg*, *Int. Ser. Monogr. Earth Sci.*, vol. 1, edited by H. Benioff, M. Ewing, B. F. Howell Jr., and F. Press, pp. 1–16, Pergamon, New York, 1958.

Båth, M., Lateral inhomogeneities of the upper mantle, *Tectonophysics*, 2, 483–514, 1965.

Casella, G., and R. L. Berger, *Statistical Inference*, Wadsworth, Belmont, Calif., 1990.

Console, R., and M. Murru, A simple and testable model for earthquake clustering, *J. Geophys. Res.*, 106, 8699–8711, 2001.

Evison, F., Multiple earthquake events at moderate-to-large magnitudes in Japan, *J. Phys. Earth*, 29, 327–339, 1981.

Evison, F., and D. Rhoades, The precursory earthquake swarm in New Zealand: Hypothesis tests, *N. Z. J. Geol. Geophys.*, 36, 51–60, 1993.

Evison, F., and D. Rhoades, Model of long-term seismogenesis, *Ann. Geophys.*, 44, 81–93, 2001.

Feller, W., *An Introduction to Probability Theory and its Applications, 2*, John Wiley, New York, 1966.

Frohlich, C., and S. D. Davis, Teleseismic b values or much ado about 1.0, *J. Geophys. Res.*, 98, 631–644, 1993.

Guo, Z., and Y. Ogata, Statistical relations between the parameters of aftershocks in time, space, and magnitude, *J. Geophys. Res.*, 102, 2857–2873, 1997.

Kagan, Y. Y., VAN earthquake predictions: An attempt at statistical evaluation, *Geophys. Res. Lett.*, 23, 1315–1318, 1996.

Knopoff, L., Y. Y. Kagan, and R. Knopoff, b-values for fore- and aftershocks in real and simulated earthquake sequences, *Bull. Seismol. Soc. Am.*, 72, 1663–1676, 1982.

Lombardi, A. M., Probabilistic interpretation of the Bath's law, *Ann. Geophys.*, 45, 455–472, 2002.

Papazachos, B. C., On certain aftershock and foreshock parameters in the area of Greece, *Ann. Geophys.*, 27, 497–515, 1974.

Purcaru, G., On the statistical interpretation of the Båth's law and some relations in aftershock statistics, *Geol. Inst. Tech. Ec. Stud. Geophys. Prosp. Buchar.*, 10, 35–84, 1974.

Reasenberg, P., Second-order moment of central California seismicity, 1969–1982, *J. Geophys. Res.*, 90, 5479–5495, 1985.

Richter, C. F., *Elementary Seismology*, 768 pp., Freeman, San Francisco, Calif., 1958.

Shi, Y., and B. A. Bolt, The standard error of the magnitude–frequency b-value, *Bull. Seismol. Soc. Am.*, 72, 1677–1687, 1982.

Tsapanos, M. T., Spatial distribution of the difference between the magnitudes of the main shock and the largest aftershock in the Circum-Pacific Belt, *Bull. Seismol. Soc. Am.*, 80, 1180–1189, 1990.

Utsu, T., Magnitude of earthquakes and occurrence of their aftershocks, *J. Seismol. Soc. Jpn.*, 10, 35–45, in Japanese, 1957.

Utsu, T., A statistical study on the occurrence of aftershocks, *Geophys. Mag.*, 30, 521–605, 1961.

Utsu, T., A method for determining the value of *b* in a formula log *n* = *a* − *bM* showing the magnitude–frequency relation for earthquakes, *Geophys. Bull. Hokkaido Univ.*, *13*, 99–103, in Japanese with English summary, 1965.

Utsu, T., Some problems of the frequency distribution of earthquakes in respect to magnitude, in Japanese with English summary, *Geophys. Bull. Hokkaido Univ.*, *17*, 85–112, 18, 53–69, 1967.

Utsu, T., Aftershock and earthquakes statistics (I), *J. Fac. Sci. Hokkaido Univ.*, *3*, 129–195, 1969.

Vere-Jones, D., A note on the statistical interpretation of Båth's Law, *Bull. Seismol. Soc. Am.*, *59*, 1535–1541, 1969.

Vere-Jones, D., Stochastic models for earthquake sequences, *Geophys. J. R. Astron. Soc.*, *42*, 811–826, 1975.

———————————

R. Console, A. M. Lombardi, and M. Murru, Istituto Nazionale di Geofisica e Vulcanologia, via di Vigna Murata, 605, I-00143 Rome, Italy. (Console@ingv.it; Lombardi@ingv.it; Murru@ingv.it)

D. Rhoades, Institute of Geological and Nuclear Sciences, P.O. Box 30-368, Lower Hutt, New Zealand. (D.Rhoades@gns.cri.nz)